SIAM J. ALG. DISC. METH.
Vol. 2, No. 4, December 1981

0196-5212/81/0204-0005 $01.00/0

COVERING REGIONS BY RECTANGLES*

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Abstract. A *board* $\mathcal{B}$ is a finite set of unit squares lying in the plane whose corners have integer coordinates. A *rectangle* of $\mathcal{B}$ is a rectangular subset of $\mathcal{B}$ and an *antirectangle* is a set of squares in $\mathcal{B}$ no two of which are in a common rectangle. We prove a conjecture of Chvátal that if $\mathcal{B}$ is convex in the horizontal and vertical directions, then the minimum number of rectangles whose union is $\mathcal{B}$ equals the maximum cardinality of an antirectangle. Our proof uses two analogous minimax theorems about covering the corners and covering the edges of the board.

We quote examples that illustrate the necessity of the hypotheses, and give some conjectures and open questions. The method of proof can give a polynomial running time algorithm for finding a minimum cover.

1. Introduction. Consider the plane covered by the unit squares whose sides lie on the integer coordinate lines; we refer to these throughout as *squares*. A *board* $\mathcal{B}$ of size n is a (finite) set of n squares. A *rectangle* (in $\mathcal{B}$ unless otherwise indicated) is a subset of $\mathcal{B}$ whose union is rectangular. A *whole cover* of $\mathcal{B}$ is a collection of rectangles whose union equals $\mathcal{B}$. The rectangles of a cover may overlap, but each of them must be wholly contained in the board. An *antirectangle* in $\mathcal{B}$ is a set of squares in $\mathcal{B}$ no two of which are contained in any rectangle. Any cover must contain at least as many rectangles as any antirectangle has squares. Therefore, if θ is the number of rectangles in a cover (the size of the cover) and α is the number of squares in an antirectangle (the size of the antirectangle) then $\theta \geqq \alpha$. We call a cover optimal if it has minimum size and an antirectangle optimal if it has maximum size. If a board has a cover and an antirectangle of equal size, then they are both optimal.

The problem of finding optimal covers and antirectangles is an example of a dual pair of packing and covering problems, well known in combinatorics (see, for example, Liu [5], Brualdi [2], Woodall [6] for more details). Chvátal originally conjectured that the optimal θ and α were equal. In general, this is false (see § 3). Here we prove his weakened conjecture that there is equality when $\mathcal{B}$ is *convex*: Whenever two squares in $\mathcal{B}$ are on the same horizontal or vertical line, all squares between them are in $\mathcal{B}$. This problem arose as an idealized special case of an operation used by the microelectronics industry. A layer of an integrated circuit (consisting of arbitrary polygons) is to be printed on a photographic plate that will become a photolithographic mask in the manufacture of the integrated circuit. The printing is done by flashing rectangles onto the photographic plate to produce an image equal to their superposition; this is to be done using as few rectangles as possible. In the "real world problem" there are additional constraints, including the discreteness of the rectangles available (which limits the accuracy), not exposing a segment of a polygon boundary more than once, and computation time and program and data space limitations. In retrospect, the

* Received by the editors February 22, 1980, and in final form March 12, 1981. This research was supported in part by the Office of Naval Research under contract N00014-76-C-0366.

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theory here in part supports certain heuristics that have been used in such a program.[1] Another context (Masek [5]) is the construction of letters and other shapes on video computer terminals.

Our method of proof can be used to obtain a polynomial time algorithm for finding the optimal θ, but we omit the details. Masek [5] established that, in general, for nonconvex boards, this problem is NP-hard. This is yet another example of a combinatorial optimization problem with a min-max theorem and an efficient algorithm, and a problem without such a theorem that is NP complete.

For any subset $S \subset \mathscr{B}$, an S-cover of $\mathscr{B}$ is a collection of rectangles whose union contains S. An S-antirectangle of $\mathscr{B}$ is an antirectangle contained in S. Let $\mathscr{E} = \mathscr{E}(\mathscr{B})$ be the set of *edge* squares, that is, those with at least one side lying on the boundary of $\mathscr{B}$. Let $\mathscr{C} = \mathscr{C}(\mathscr{B})$ be the set of *corner* squares, those with two adjacent sides on the boundary. Nonedge squares are called *interior* squares. Our proof of the main theorem relies on induction for certain boards (called *reducible*). For the remaining (*irreducible*) boards, the proof uses an analogous min-max result for edge covers and edge antirectangles, which holds for these boards. The proof of this edge result makes use of a theorem about corner covers and antirectangles.

These problems can be restated in familar graph theoretic terminology by associating a board $\mathscr{B}$ with a graph $G = G(\mathscr{B})$ whose nodes are the squares in $\mathscr{B}$ and in which two squares are joined by an arc if there is a rectangle in $\mathscr{B}$ that contains them both. The following simple lemma, which is true for any board, is presented without proof.

LEMMA 1.1 *The cliques of $G(\mathscr{B})$ are the rectangles of $\mathscr{B}$.*

Let S be any subset of $\mathscr{B}$ and let G_S denote the induced subgraph of G on this subset. A maximum S-antirectangle of $\mathscr{B}$ is, by definition, a maximum independent set of G_S, whose size is written $\alpha(G_S)$. A minimum S-cover of $\mathscr{B}$ has size equal to $\theta(G_S)$, the minimum number of cliques needed to cover G_S.

Our main results, which hold for convex boards $\mathscr{B}$, are:

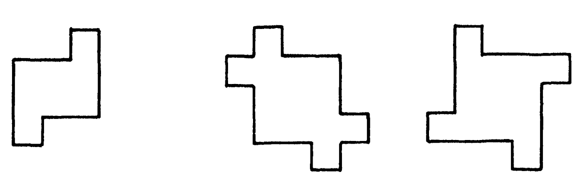
THEOREM I. *The minimum size of a corner cover of $\mathscr{B}$ equals the maximum size of a corner antirectangle in $\mathscr{B}$; i.e., $\theta(G_{\mathscr{C}}) = \alpha(G_{\mathscr{C}})$.*

LEMMA. 6.5. *If $\mathscr{B}$ is irreducible (see § 5), then the minimum size of an edge cover of $\mathscr{B}$ equals the maximum size of an edge antirectangle in $\mathscr{B}$, i.e., $\theta(G_{\mathscr{E}}) = \alpha(G_{\mathscr{E}})$.*

THEOREM II. *The minimum size of a whole cover of $\mathscr{B}$ equals the maximum size of an antirectangle in $\mathscr{B}$, i.e., $\theta(G) = \alpha(G)$.*

It is shown (§ 3) that the convexity hypothesis in these results is necessary. Figure 1 illustrates that the sizes of the optimal corner, edge, and whole covers may be different.

Theorem I is proven by showing that $G_{\mathscr{C}}$ has a simple structure.



No optimal corner cover covers all the edges.

No optimal edge cover covers the whole board.

FIG. 1

[1] The first author encountered this problem during his employment with Applicon, Inc.

LEMMA 4.1. *Each connected component of $G_{\mathscr{C}}$ is either*

(1) *a 4 clique, or*

(2) *a graph in which every odd cycle contains a square of degree 2, whose neighbours are adjacent.*

To prove Lemma 6.5, we first show that irreducibility implies that in a corner cover by maximal rectangles, the rectangle covering a corner covers all of the squares on at least one of the edges incident to the corner. This fact enables us to construct an arc-deleted subgraph G^* of $G_{\mathscr{C}}$ which satisfies $\alpha(G^*) = \theta(G^*)$ and whose independent sets and clique covers correspond to $\mathscr{E}$-antirectangles and $\mathscr{E}$-covers.

Theorem II is proved in two steps. First, two *reducible configurations* are defined. Each configuration involves a maximal rectangle that must be in every optimal cover, and a reduction which produces a smaller board. The reduction is such that from an equally sized cover and antirectangle pair for the reduced board (obtained by induction), we can construct an equally sized pair for the original board. The second step involves analysis of irreducible boards. If some optimal edge cover covers the whole board, Lemma 6.5 implies the result. If not, we show the board is so structured that an optimal edge cover that covers a maximal set of squares provides an equally sized antirectangle. To these, one more rectangle and one more square can be added to yield an optimal whole cover and antirectangle.

In the next section are definitions and simple facts used in the rest of the paper. Section 3 gives examples that indicate the necessity of the convexity hypothesis, and some open problems. The remainder of the paper is restricted to convex boards. Section 4 is about corner covering. Section 5 gives the reducible configurations. Section 6 presents the edge covering result. Finally, in § 7 we finish covering the whole board when it is irreducible.

2. Definitions and simple facts. We adopt the usual coordinate system in which the positive x axis points to the *right* and the y axis points *up*. The defines for us the everyday words for direction: top, below, horizontal, etc. The squares are the unit squares bounded by integer coordinate lines. We identify a board or rectangle (set of squares) with its union (a polygon). A board has *vertices* and *edges* on its boundary, but the edges of a rectangle are called *sides*.

A vertex with interior included angle of 90° is called a *corner vertex*; a *corner square* is one that touches a corner vertex. The integer coordinates divide each edge into unit (boundary) *segments*. A square that is bounded on at least one side by a segment is called an *edge square*. Every corner vertex is associated to a unique corner square and every segment is associated to a unique edge square. Let $\mathscr{C}$ and $\mathscr{E}$ denote, respectively, the sets of corner and edge squares.

Points, such as vertices, are referred to by their coordinates (x, y). We make the special convention that the cordinates of a square or segment are those of its center. The coordinates of z are denoted by $x(z)$ and $y(z)$. When z is an edge, $t \geqq x(z)$ means $t \geqq x(p)$ for all points p in z. This way, for example, we say square u is to the right of vertical edge CD and left of vertical edge EF by $x(CD) < x(u) < x(EF)$. Occasionally, $x \pm 0.5$ is used to convert between coordinates of points and those of squares or segments.

A rectangle is said to *cover* a corner, a square or an edge if it contains the corresponding corner square, the indicated square, or all the edge squares on the edge respectively.

An edge is called a *support edge* when both its ends are corner vertices. It is easy to see every support edge has a unique maximal rectangle that covers it. This is called the *associated* or *support edge rectangle.* Any rectangle can be named by giving either a point, segment, or square on two opposite corners. Thus $\square uv$ denotes the smallest rectangle that contains u and v.

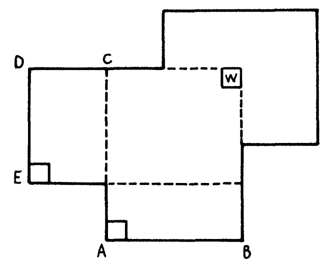


FIG 2. *$\square BC$ is the support edge rectangle of AB; $\square EW$ is the support edge rectangle of DE.*

$G=G(\mathcal{B})$ is a graph whose nodes, called squares, are the squares in $\mathcal{B}$. Two squares s and t are joined by an arc, denoted $(s,t)\in G$, whenever $\square st\subseteq\mathcal{B}$. An easy way to check whether $(s,t)\notin G$ is to check whether the interior of $\square st$ meets a boundary segment or a square not in $\mathcal{B}$. The neighborhood of a square s, $N_G(s)=N(s)$, is the set of all squares that can be covered by some rectangle covering s:

$$N_G(s)=\{t\in\mathcal{B}|(s,t)\in G\}\cup\{s\}$$

Let R be the associated rectangle of support edge AB. The side of R opposite AB must contain a boundary segment. Let e be the edge square on AB that meets the perpendicular bisector of that segment. R is the unique maximal rectangle that covers e. Furthermore, in any cover and antirectangle problem in which e must be covered, there is always an optimal cover that contains R and an optimal antirectangle that contains e.

All of our positive results concern *convex* boards as defined in § 1. In other words, for any $s_1, s_2\in\mathcal{B}$, if $x(s_1)=x(s_2)$ or $y(s_1)=y(s_2)$ then $\square s_1s_2\subseteq\mathcal{B}$. We make repeated use of the following facts about convex $\mathcal{B}$:

Fact 2.1. Given a pair of squares, $s_1, s_2\in\mathcal{B}$, consider the other two squares at the corners of $\square s_1s_2$, $s_1'=(x(s_1), y(s_2))$, $s_2'=(x(s_2), y(s_1))$. When (s_1, s_2) is in G, then s_1' and s_2' are in $\mathcal{B}$. Convexity means that the converse is true. Then, we need to look at only two squares, s_1' and s_2', to see whether $(s_1, s_2)\in G$.

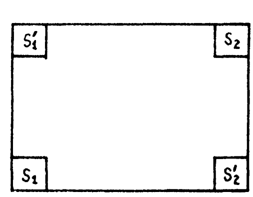


FIG. 3. *Sufficient condition for $(s_1, s_2)\in G$ for convex $\mathcal{B}$.*

Fact 2.2. A convex board has exactly 4 support edges. These divide the boundary into 4 support edges and 4 possibly empty paths.

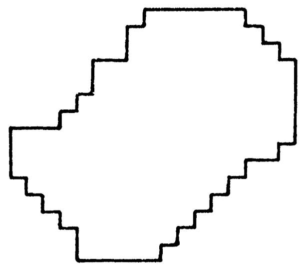


FIG. 4. *Note the four support edges of a convex board.*

The arguments made in succeeding sections hold under rotations and reflections of the board. For simplicity, the arguments are described and illustrated with the board in a specific position.

3. Counterexamples and open questions. Chvátal's original conjecture was disproved by Szemerédi who found a counterexample with a "hole" (Fig. 5). Chung (who informed us of the history of this problem) then found the simply connected counterexample in Fig. 6 (Chung [3]).

One can see that optimal θ and α in these examples are unequal by first observing that a support edge rectangle R always contains some edge square such that R is the unique maximal rectangle that contains that square. Thus one can assume that the support edge rectangles are all in the optimal cover and that one edge square from each is in the optimal antirectangle. Second, consider the squares S left uncovered by these rectangles (not cross hatched in the drawings). In each example, G_S has an induced 5-cycle (indicated by connected dots in the drawings). Hence at least 3 cliques are needed to cover S. One can verify that 3 cliques suffice. Finally, one can verify there are only up to two independent squares in S.

Similar analyses of Figs. 7 and 8 yield 7-cycles, and so show that the corner covering result ($\theta(G_{\mathscr{C}}) = \alpha(G_{\mathscr{C}})$) and the edge covering result ($\theta(G_{\mathscr{E}}) = \alpha(G_{\mathscr{E}})$) are sometimes false for nonconvex boards.

A graph G is perfect if for all subsets S of vertices, $\theta(G_S) = \alpha(G_S)$ (see Berge [1]). Figure 9 shows a board with a subset of squares that includes a 5-cycle in G; hence $G(\mathscr{B})$ is not always perfect, even for convex boards.

In this paper we show that $\theta(G_{\mathscr{E}}) = \alpha(G_{\mathscr{E}})$ for certain (irreducible) convex boards (§ 6). One of the authors, M. Saks, has recently shown that for any convex board, and any subset $\mathscr{E}'$ of edge squares, $\theta(G_{\mathscr{E}'}) = \alpha(G_{\mathscr{E}'})$; hence $G_{\mathscr{E}}$ is a perfect graph.

We have not yet fully investigated the implications of convexity in only one direction.

Finally, for arbitrary boards $\mathscr{B}$, let θ and α be the optimal cover and antirectangle sizes respectively. Erdös asked if θ/α is bounded and we do not know the answer. Chung's example has $\theta/\alpha = \frac{8}{7}$. The largest value we could achieve for θ/α is $\frac{21}{17} - \varepsilon$,

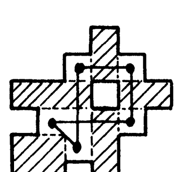


FIG. 5. $\theta(G) = 8$, $\alpha(G) = 7$.

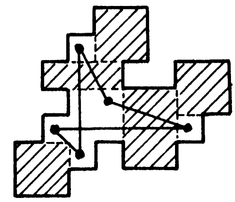


FIG. 6. $\theta(G)=8$, $\alpha(G)=7$.

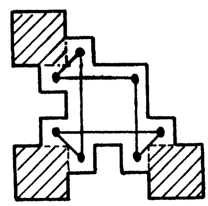


FIG. 7. $\theta(G_{\mathscr{C}})=7$, $\alpha(G_{\mathscr{C}})=6$.

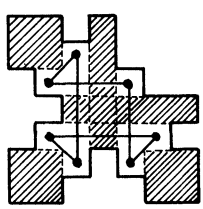


FIG. 8. $\theta(G_{\mathscr{C}})=9$, $\alpha(G_{\mathscr{C}})=8$.

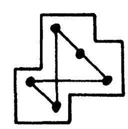


FIG. 9. *Induced 5-cycle in G.*

4. Corner covering. A corner vertex of the board is designated as type (left, upper), (left, lower), (right, upper), or (right, lower) according to its position with respect to the board square it touches. A corner square has the types of its incident corner vertices (it may have one or two for nontrivial boards). The types of two corners c_1, c_2 provide necessary conditions for $(c_1, c_2) \in G$.

If c_1, c_2 have a common type, $(c_1, c_2) \notin G$ (Fig. 11).

If c_1, c_2 each have one type, and the types differ in both components, we say c_1, c_2 have opposite types. Whether $(c_1, c_2) \in G$ or not depends on the rest of $\mathscr{B}$ (Fig. 12).

If c_1, c_2 have types that differ in one component (say the x, i.e., "left", "right" component) and agree in the other, we say they have adjacent type. In this case, $(c_1, c_2) \in G$ only if c_1 and c_2 have equal coordinates in the component in which their types agree (say the y component). This condition is sufficient when $\mathscr{B}$ is convex (Fig. 13).

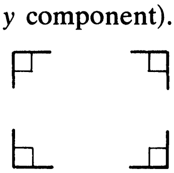


FIG. 10. *The four corner types.*

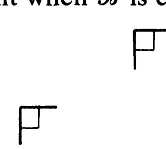


FIG. 11. *Corners with common type.*

FIG. 12. *Corners with opposite type.*

FIG. 13. *Corners with adjacent type.*

LEMMA 4.1. *For a convex board β, each connected component of $G_{\mathscr{C}}$ is either*

(1) *a 4 clique, or*

(2) *a graph in which every odd cycle contains a square of degree 2 in $G_{\mathscr{C}}$ such that its two neighbors are adjacent (thus these three vertices from a 3 clique).*

Proof. Let $(c_1, \cdots, c_n)$, $n \geqq 3$, be a odd cycle in $G_{\mathscr{C}}(\mathscr{B})$. Then at least two successive squares in the cycle, say c_1 and c_2, are of adjacent type.

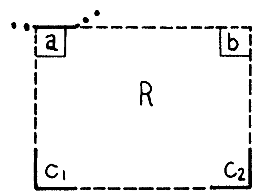


FIG. 14

There is a unique maximal rectangle R that contains both c_1 and c_2. Convexity implies the side of R opposite c_1 and c_2 meets a boundary segment at at least one of its ends; assume that end is closest to c_1 (Fig. 14). Therefore, $R = N_G(c_1)$. Thus in $G_{\mathscr{C}}$, c_1 is connected only to c_2 and to whichever of a and b are corners. If both a and b are corners, c_1, c_2, a, b is a 4 clique that is not connected to any other corner (Fig. 10). If only one of a and b, say a, is a corner, then c_1 has degree 2 and a, c_1, c_2 is a 3-clique. If neither is a corner, then c_1 is not contained in a cycle.

THEOREM I. *The minimum size of a corner cover of $\mathscr{B}$ equals the maximum size of a corner antirectangle in $\mathscr{B}$.*

Proof of Theorem I. We show that any graph satisfying the properties proved in Lemma 4.1 has a clique cover and independent set of equal size. The cover consists of all of the (disconnected) 4-cliques, all 3-cliques, and a minimum cover of the subgraph H obtained by deleting all of these cliques. The independent set consists of one square from each 4-clique, a degree 2 square from each 3-clique and a maximum independent set in H. (This set is independent since every square in H is independent of the degree 2 squares in each 3-clique.) Now H is bipartite, i.e., it has no odd cycles, since otherwise H would contain a 3-clique by Lemma 4.1. It is a well-known consequence of the König–Egerváry theorem that the size of the minimum clique cover of a bipartite graph equals that of the largest independent set. Thus the given cover and independent set of $G_{\mathscr{C}}$ have the same size. □

In subsequent sections, we will need the following stronger result.

LEMMA 4.2. *Say H is a subgraph of $G_{\mathscr{C}}$ obtained by deleting some arcs such that whenever the arc joining the two neighbors of a degree 2 square s of a 3-clique is deleted, an arc incident to s is also deleted. Then $\alpha(H) = \theta(H)$.*

Proof. It is easy to see that deleting any such arcs preserves the property of $G_{\mathscr{C}}$ proved in Lemma 4.1 and thus the proof of Theorem I applies.

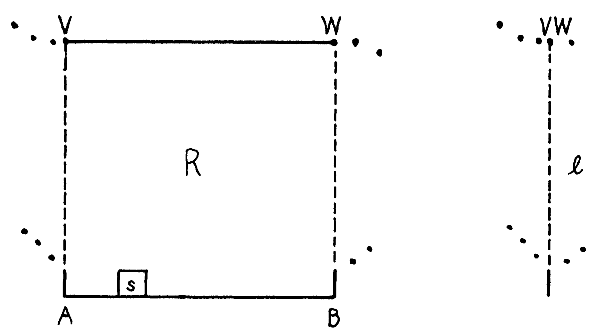
5. Reducible configurations. Assume $\mathscr{B}$ is a convex board.

THEOREM II. *The minimum size of a whole cover of $\mathscr{B}$ equals the maximum size of an antirectangle in $\mathscr{B}$.*

This section comprises the part of the proof of Theorem II that treats some boards by induction on the number of squares. Such boards, which are called *reducible*, have "reducible configurations". A reducible configuration cannot occur in a smallest counterexample to the theorem. Each reducible configuration is accompanied by two constructions. The first produces a smaller "reduced" board $\mathscr{B}'$ from the reducible board $\mathscr{B}$. The second produces an optimal cover and antirectangle for $\mathscr{B}$ from such a pair for $\mathscr{B}'$. We have two reducible configurations. Each involves a support edge and its associated rectangle. Throughout, assume the relevant support edge is the bottom one.

Tab reduction. Let R be the rectangle associated with a support edge. If the side of R opposite the support edge lies entirely on one edge of the board then $\mathscr{B}$ has a *tab reduction* (Fig. 15a).

$\mathscr{B}'$ is constructed by deleting all the squares in R. In other words, the top and bottom sides of R are collapsed to points and then all pairs of vertical segments that now coincide are deleted. Clearly, $\mathscr{B}'$ is convex. Assume the undeleted squares in $\mathscr{B}$ retain their identity in $\mathscr{B}'$. R itself is collapsed to vertical line l in $\mathscr{B}'$ (FIG. 15b).



(a) (b)

FIG. 15. *Tab reducible configuration before and after reduction.*

From a cover of $\mathscr{B}'$ it is easy to construct a cover of $\mathscr{B}$. Take R and all the rectangles from the former, after stretching horizontally any that cross l. From an antirectangle in $\mathscr{B}'$ construct one in $\mathscr{B}$ by taking all squares in the former and adding any edge square s on the bottom support. The antirectangle in $\mathscr{B}'$ is an antirectangle in $\mathscr{B}$ and remains such when s is added because $N_G(s) = R$ and $\mathscr{B}' = \mathscr{B} \cup R$.

This construction gives an optimal pair in β given an optimal pair for $\mathscr{B}'$; by induction the optimal sizes for $\mathscr{B}'$ are equal, and so they are for $\mathscr{B}$.

Partial tab reduction. For convenience, the conditions for a partial tab reduction are stated for AB the bottom, and GH the right support edge. They can apply to any perpendicular pair of supports.

Conditions for partial tab reduction of $\mathscr{B}$ at AB:

Condition 5.0. $\mathscr{B}$ has no tab reduction.

Condition 5.1. All points in the rectangle associated with GH lie strictly to the right of AB (Fig. 16).

In other words, (5.0) means that the side opposite the support edge of every support edge associated rectangle does not lie entirely on the edge it meets. Condition (5.1) can be restated as $x(B) < x$(left side of rectangle associated with GH) if we assume $x(A) < x(B)$.

Let $R = ABWV$ be the rectangle associated with AB. We examine the consequences of the no tab reduction hypothesis.

First, the top of R cannot be contained in an edge (Fig. 15). Neither can it cover horizontal segments both at its left and right sides (Fig. 17a); otherwise, there would

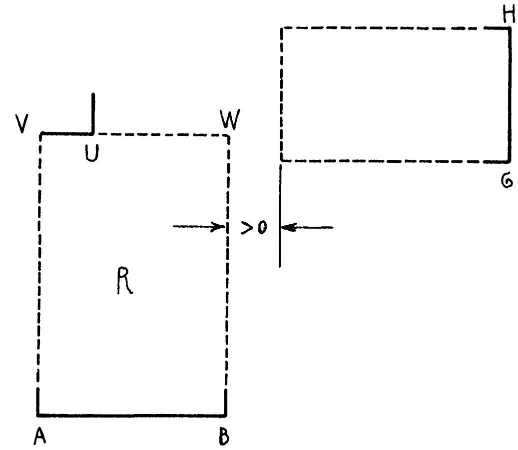


FIG. 16. *Condition for partial tab reduction. No relation between $y(V)$ and $y(G)$ is implied in this figure.*

be a tab reduction at the top support (which must then lie strictly between A and B). We assume without loss of generality that the top of R meets a horizontal edge along UV on the left (Figs. 17b,c). Note that the left support is between A and V (vertically).

Second, we claim that W (upper right corner of R) cannot touch the boundary at all. For if its does, either there is a tab reduction at the top support (Fig. 17b), or BW is part of an edge (Fig. 17c). In the latter case, there is a tab reduction at the left support.

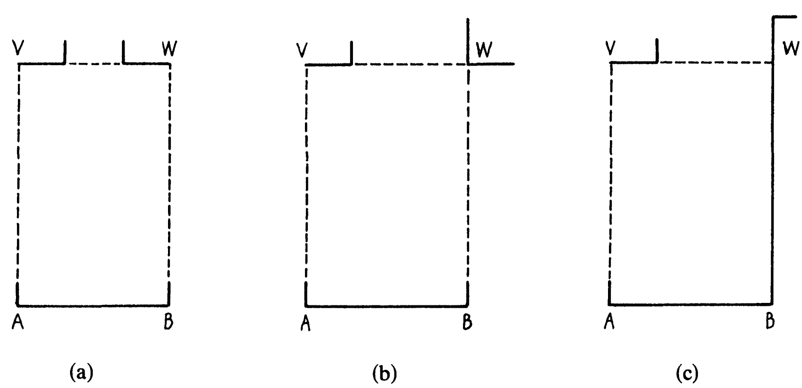


FIG. 17. *Cases eliminated by no tab reduction condition in condition for partial tab reduction.*

We conclude that part of the board looks something like Fig. 18a. The top support does not lie between $x = x(U)$ and $x = x(B)$. BC is the vertical edge at B and AD is the vertical edge at A (D may equal V). Let $y = \min(y(C), y(D))$. $\mathscr{B}'$ is constructed by deleting all squares in $\square A(x(B), y) \cup \square AU$ (Figure 18b). This is equivalent to collapsing all vertical segments with y coordinate between $y(A)$ and y; and all horizontal segments with x coordinate between $x(A) = x(V)$ and $x(U)$. Let $Q = (x(U), y)$.

The construction of a cover for $\mathscr{B}$ from one for $\mathscr{B}'$ is similar to that used for the tab reduction. Take all rectangles from the latter, after stretching any that cross line

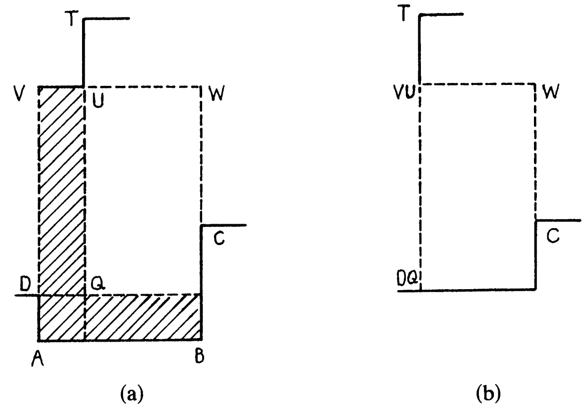


FIG. 18. *Partial tab reducible configuration before and after reduction.*

QU, (which $\square AU$ was collapsed to), and add R. In this cover for $\mathcal{B}$, note $\square QW$ is covered by R and at least one other rectangle. (In Fig. 19 there is a board with no tab reductions and in which Condition 5.1 fails. In it, the analogue of $\square QW$ contains square d which is covered only once in the unique minimum cover. Thus Condition 5.1 is a necessary hypothesis for the above construction to yield an optimal cover.)

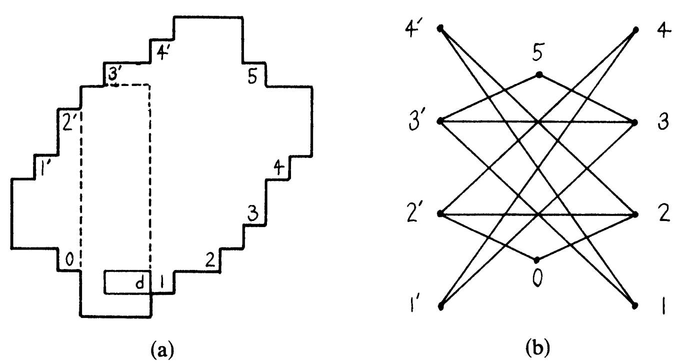


FIG. 19. *Example illustrating the necessity of the partial tab reduction condition* (*or something like it*) *for the construction of an optimal cover of β from that of the reduced board.* (a) *has a unique optimal cover consisting of the support edge associated rectangles and* $\square 33'5$, $\square 22'0$, $\square 1'4$ *and* $\square 14'$. *This can be seen from* (b) *which is the induced subgraph of G on the squares not covered by the support edge rectangles. Square d is in the partial tab reduced board but is covered only by the support edge rectangle shown.*

The construction of the antirectangle in $\mathcal{B}$ is more complicated. Let A also denote the corner square at A. Observe $N_G(A)=R$. Let $\mathcal{A}'$ be an antirectangle in $\mathcal{B}'$. If $\mathcal{A}'\cup\{A\}$ is an antirectangle, which means $\mathcal{A}'$ has no square in $\square QW$, we are done. Otherwise $\mathcal{A}'\cap\square QW=\{p\}$ and we claim we can replace p in $\mathcal{A}'$ by some other square to produce a set that remains an antirectangle when A is added to it.

Case 1. $y(D)=y(Q)<y(p)<y(C)$, in other words, Q is below C and $p\in\square QC$. Let f be the leftmost square in $\mathcal{B}$ with $y(f)=y(p)$. The hypothesis and the board structure (convexity) imply that f is left of A and $N_G(f)\subseteq N_G(p)$. Hence $\mathcal{A}=\mathcal{A}'\backslash\{p\}\cup\{A,f\}$ is the desired antirectangle (see Fig. 20a).

Case 2. $y(p)>y(C)$. We prove we can replace p in $\mathcal{A}'$ by at least one of two other squares. Let e be the square $(x(C)+0.5, y(p))$. Move p right just beyond R.

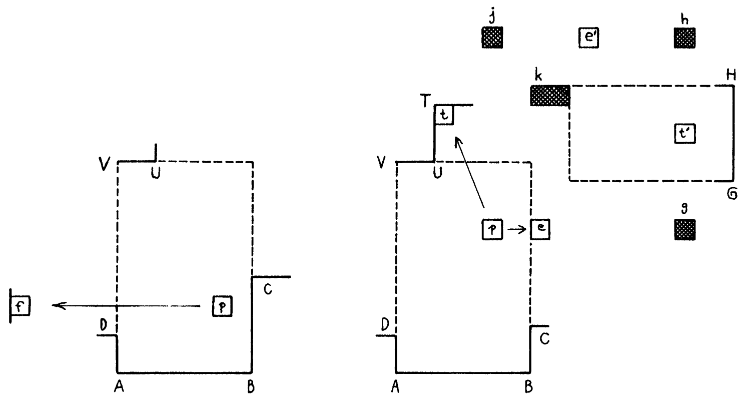


FIG. 20. *Two cases in construction of maximum antirectangles.*

Let t be the corner square at the top end of the vertical edge at U. We show the assumption that neither $\mathcal{A}'\backslash\{p\}\cup\{e\}$ nor $\mathcal{A}'\backslash\{p\}\cup\{t\}$ is an antirectangle leads to a contradiction (see Fig. 20b).

If $\mathcal{A}'\backslash\{p\}\cup\{e\}$ is not an antirectangle, then for some $e'\in\mathcal{A}'$, $(e, e')\in G$ but $(p, e')\notin G$. Since $y(p)=y(e)$, $(x(e'), y(e))\in\beta$ so $j=(x(p), y(e'))\notin\mathcal{B}$. (We use Fact 2.1.)

If $\mathcal{A}'\backslash\{p\}\cup\{t\}$ is not an antirectangle, then for some $t'\in\mathcal{A}'$, $(t, t')\in G$ but $(p, t')\notin G$. The fact that the top support extends right of $x=x(C)$ implies $x(t')>x(C)$. (For if $x(t')<x(C)$ and still $(p, t')\notin G$ and $(t, t')\in G$, then $x(t')<x(p)$ and $(x(p), y(t'))\notin\mathcal{B}$.) Therefore, $(t, t')\in G$ implies $(x(p), y(t'))\in\mathcal{B}$, and so $g=(x(t'), y(p))\notin\mathcal{B}$.

Furthermore, $y(e')>y(t)$ and $y(t')\leqq y(t)$, so $e'\neq t'$. Convexity implies $(x(e'), y(t'))\in\mathcal{B}$, so $\{e', t'\}\subseteq\mathcal{A}'$ implies $h=(x(t'), y(e'))\notin\mathcal{B}$. Some squares in $\mathcal{B}$ (including t') are between g and h, so the rectangle associated with the right support GH must pass between g and h. The condition for partial tab reduction implies that this associated rectangle cannot extend left as far as $x=x(e)$. By convexity, some square $k\notin\mathcal{B}$ has $x(k)=x(e)$, and $y(k)<y(H)<y(h)=y(e')$. This contradicts $(e, e')\in G$.

Hence, for at least one $f\in\{e, t\}$, $\mathcal{A}=\mathcal{A}'\backslash\{p\}\cup\{A, f\}$ is an antirectangle. Suppose we started with an optimal cover and antirectangle for $\mathcal{B}'$. The induction hypothesis (Theorem II) implies that the optimal sizes for $\mathcal{B}'$ are equal, and our construction increases both by one. □

6. Edge covering. In this section, we prove $\alpha(G_{\mathscr{C}}(B))=\theta(G_{\mathscr{C}}(B))$ for convex, irreducible $\mathcal{B}$. More generally, we show $\alpha(G_{\mathscr{C}})=\theta(G_{\mathscr{C}})$ for convex boards that have, at each corner, at most one incident edge that can be partially covered by a maximal rectangle. We assume all rectangles are maximal.

DEFINITION. Suppose CD is an edge, and C is a corner. We say rectangle R *partially covers CD at C* if R covers corner square C, but not every edge square on CD.

Our definition means an edge may be partially covered only at an end that is a corner (see Fig. 21).

LEMMA 6.1. *An edge cannot be partially covered at both of its ends.*

Proof. If it were, convexity would imply one of the rectangles is not maximal. □

Now, suppose edge CD is partically covered at C. Let R be the rectangle that partially covers CD at C and that covers as much of CD as possible. Then the side of R perpendicular to CD that does not touch C must cover a segment with an end point E closest to CD. The edge incident to E that is parallel to CD must reach at least as far as D (see Fig. 21).

Let c' be the edge square on CD closest to C but not in R.

Fact 6.2. Any maximal rectangle that covers c' also covers all of CD (and all of the other edge of C if the other edge cannot be partially covered at C).

Fact 6.3. $N_G(C) \supseteq N_G(c')$.

We call c' a "proxy" for C.

LEMMA 6.4. *Let $\mathscr{B}$ be a convex board for which at each corner there is at most one incident edge that can be partially covered at that corner. Then* $\alpha(G_{\mathscr{C}}(\mathscr{B})) = \theta(G_{\mathscr{C}}(\mathscr{B}))$.

Proof. Let $\mathscr{P} \subseteq \mathscr{B}$ consist of the unique proxy for each corner square at which an edge can be partially covered, plus all corner squares at which neither incident edge can be partially covered. Fact 6.3 implies that $G^* = G_{\mathscr{P}}(\mathscr{B})$ is isomorphic to an arc deleted subgraph of $G_{\mathscr{C}}$. We claim this subgraph of $G_{\mathscr{C}}$ satisfies the hypothesis of Lemma 4.2. Consider any 3 clique $T = \{a, b, c\}$ in $G_{\mathscr{C}}$. If $b \in T$ is the square at the right angle of a triangle of squares, that is, the degree 2 square of Lemma 4.2, it cannot be a corner square that is replaced by a proxy. On the other hand, let $c \in T$ be a corner square that is replaced by a proxy c'; then neither (a, c') nor (b, c') is an arc in G^*, or else both are. Therefore, Lemma 4.2 implies $\theta(G^*) = \alpha(G^*)$. Fact 6.2 implies any set of maximal rectangles corresponding (using Lemma 1.1) to a clique cover of G^* covers all the edges. □

LEMMA 6.5. *If β is convex and irreducible,* $\alpha(G_{\mathscr{C}}(\mathscr{B})) = \theta(G_{\mathscr{C}}(\mathscr{B}))$.

Proof. Suppose the hypothesis for Lemma 6.4 were false for $\mathscr{B}$. If $\mathscr{B}$ were as in Fig. 22, there would be a tab reduction between U and V. Hence $\mathscr{B}$ must be as in Fig. 23.

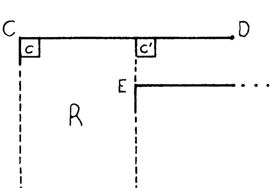


FIG. 21

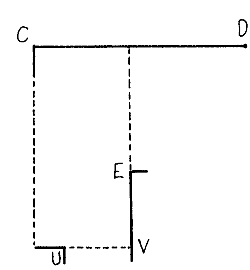


FIG. 22

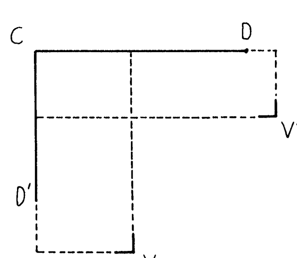


FIG. 23

Since there are no tab reductions in this $\mathscr{B}$, (by convexity) some of the right support edge must be above $y = y(D)$. Hence, the left side of the right support edge associated rectangle R cannot be left of $x = x(D)$. Let B be the right end of the bottom support. $x(B) < x(V)$ (again by convexity) and $x(V) < x(D)$, so B is strictly left of the left side of R. This is Condition 5.1 for partial tab reduction at the bottom support, and so by contradiction, the hypothesis of Lemma 6.4 is true. □

7. Whole covering of irreducible boards. This section concludes the proof of Theorem II. We prove that if every optimal edge cover of an irreducible board $\mathscr{B}$ fails to cover every square, then an optimal whole cover can be obtained by adding one rectangle to an optimal edge cover. Suppose $\mathscr{B}$ is such a board. Consider an optimal edge cover **C** (by maximal rectangles) that covers the maximum number of squares. Let z be an uncovered square.

There are three steps. The first is to establish the structure of $\mathscr{B}$. The second is to show the squares not covered by **C** can all be covered by one rectangle. The third is to prove optimality by constructing an antirectangle that contains z and one square for each rectangle of **C**.

Step 1. *Structure of* $\mathscr{B}$. In each of the four directions, there is an edge square on the same line as z. Let these squares be r_i, r'_i, $i = 1, 2$ as shown in Figs. 24–25. Consider

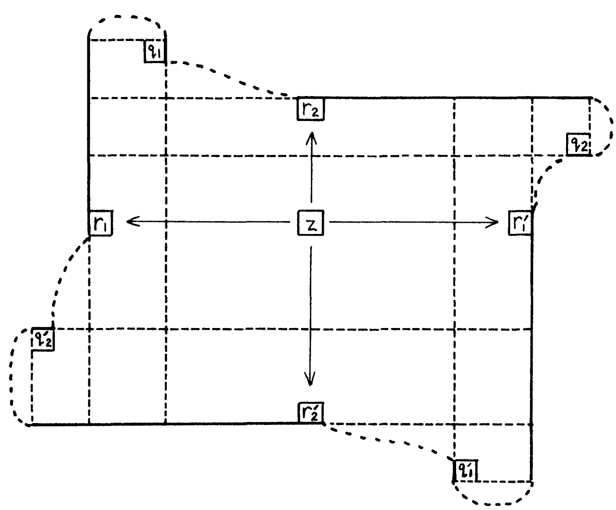


FIG. 24. *Dotted lines indicate polygon boundary schematically. In this case, the board is reducible.*

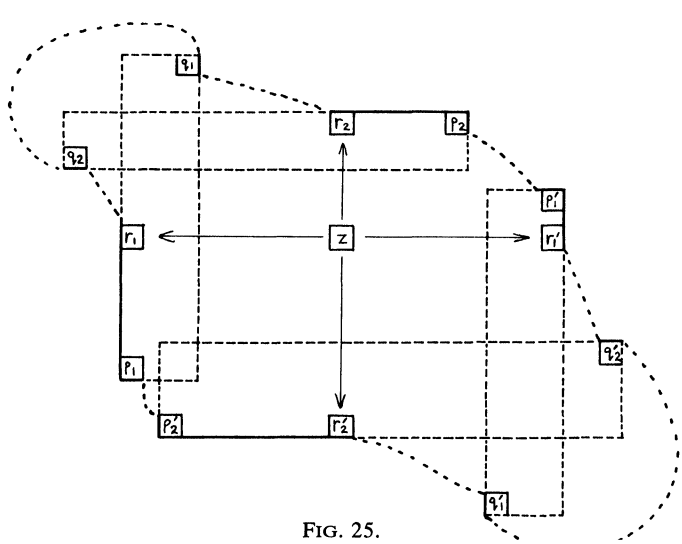


FIG. 25.

a pair of rectangles in **C** that cover r_1 and r_1'. We can assume without loss of generality that the "inner" (right) side of the (left) rectangle covering r_1 meets a vertical segment q_1 at its upper end, and the "inner" (left) side of the (right) rectangle covering r_i' meets a segment q_1' at its lower end. Each inner side must meet a segment because the rectangles are maximal. The two parallel inner sides could not each meet a segment at the same end because this would violate convexity.

Apply the above argument to the rectangles that cover r_2 and r_2'. Now there are two possibilities. The first, that the "inner" (bottom) side of the (top) rectangle which covers r_2 meets a segment on the right, is shown in Fig. 24. Here, convexity implies the solidly drawn parts of the rectangle sides are parts of edges, so there is a tab reduction at each support edge; this contradicts the assumption that $\mathscr{B}$ is irreducible. Hence our four rectangles must meet the boundary of $\mathscr{B}$ as in Fig. 25. Taking into account the general structure of convex boards, we conclude:

Let $\mathbf{K}_i(\mathbf{K}_i')$ be the set of rectangles in **C** that contain squares that lie between z and $r_i(r_i')$. Each rectangle in $\mathbf{K}_i$ ($\mathbf{K}_i'$) covers two segments q_i and $p_i(q_i'$ and $p_i')$ as the rectangles shown in Figs. 25 and 28 do. Here, segments labeled q block extension of the rectangle toward z and segments labeled p block extension away from q.

For example, any rectangle in $\mathbf{K}_1$ covers a horizontal segment like p_1 at its lower left corner and a vertical segment like q_1 at its upper right corner.

The boundary segments and vertices are cyclically ordered counterclockwise (CCW). When e and f are two edge squares $[e, f]$ denotes all the edge squares on any segment on the CCW path from a segment on e to a segment of f. $[e, f) = [e, f]\backslash\{e\}$, etc.

Step 2. *Covering uncovered squares with one more rectangle.* Consider the four support edge rectangles. The result of Step 1 and the irreducibility hypothesis imply these rectangles must be as in Fig. 26. These rectangles "cross" at D and B and intersect at A and C, as shown. By convexity, $\square ABCD \subseteq \mathscr{B}$. We claim all squares not covered by **C** lie in $\square ABCD$. To prove this, we note that the nonsupport (maximal) rectangles in edge cover **C** also cover the region between the boundary and the support rectangles. For example, again looking at Fig. 26, we see that any rectangle that covers any corner square f_1 or f_2 in $[E_1, E_2]$ or $[E_4, E_5]$ covers $\square f_1D$ or $\square f_2A$ respectively.

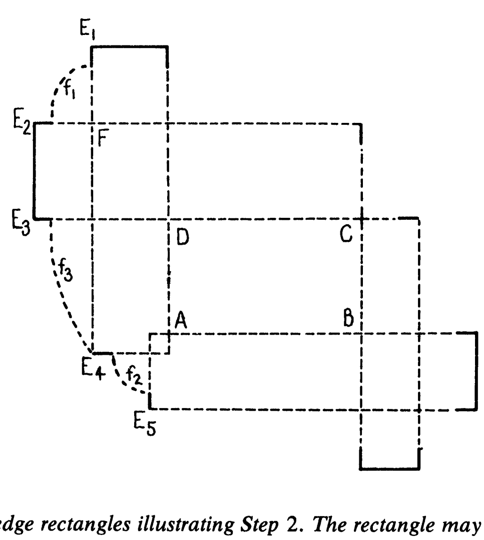


FIG. 26. *The support edge rectangles illustrating Step* 2. *The rectangle may overlap* (*as A*) *or touch* (*as C*) *at A and C. All uncovered squares must be in* $\square ABCD$.

A rectangle covering edge square f_3 in $[E_3, E_4]$ covers all the squares on the horizontal line from f_3 to DA.

Step 3. *Constructing maximum antirectangles.*

DEFINITION. Let $\mathbf{C}$ be a set of rectangles. A square $s \in \mathcal{B}$ is *critically covered* when it is in only one rectangle in $\mathbf{C}$. Two squares are *matched* by $R \in C$ if both are critically covered by R.

We note that $\mathbf{C}$ is a minimum edge cover, so every rectangle in $\mathbf{C}$ critically covers at least one edge square. If $\mathcal{A}$ is an antirectangle and $|\mathcal{A}| = |\mathbf{C}|$, $\mathcal{A}$ consists only of critically covered squares.

Let us choose one rectangle from each $\mathbf{K}_i$, $\mathbf{K}_i'$ that is closest to square z from among all rectangles in $\mathbf{K}_i$, $\mathbf{K}_i'$. These four rectangles and the two boundary segments p, q on each we described in Step 1 and are displayed in Fig. 28. Let $\mathbf{K} = \mathbf{K}_1 \cup \mathbf{K}_2 \cup \mathbf{K}_1'\mathbf{K}_2'$. We divide the proof into numbered assertions.

1) If an edge square $e \in [s_1, s_2] \cup [s_1', s_2']$ on edge E is critically covered, so is the corner square c on E. For the maximal rectangle covering c is unique and contains e. Let S be the set of corner squares in $[s_1, s_2] \cup [s_1', s_2']$.

2) Let $\bar{z} = \{s | s$ is an edge square and $(z, s) \notin G\}$. We claim that every rectangle in $\mathbf{K}$ critically covers a square in $\bar{z}$. Every rectangle R in, for example, $\mathbf{K}_1$ critically covers some edge squares. If R critically covers some squares in (r_2, r_1) we are done. Otherwise, R must critically cover some edge squares $E \subseteq [r_1, r_2')$ and no edge squares elsewhere. Now, if R did not critically cover any interior squares above line $r_1 z$, we could replace R in $\mathbf{C}$ by a maximal rectangle R_1 obtained as follows. Shrink the top of R down to $r_1 z$, then extend the right side as far as possible, and then finally extend the top to make the rectangle maximal. The result is a minimum edge cover that covers more squares than $\mathbf{C}$. On the other hand, suppose R critically covered a set of interior squares H above line $r_1 z$. The same replacement process for R still produces a minimum edge cover that covers z. If this new edge covers H, again the maximality of $\mathbf{C}$ is contradicted. If not, the uncovered squares in H are squares not covered by a minimum edge cover of maximal rectangles and so they lie (along with z) in $\square ABCD$ as we showed in Step 2. Therefore, there exists a rectangle R_2 covering $E \cup H \cup \{z\}$ (see Fig. 27).

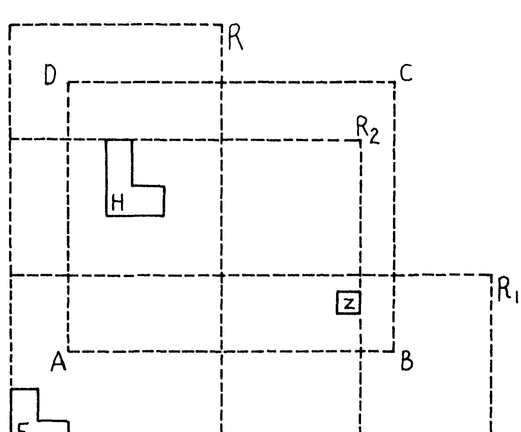


FIG. 27. *The maximality of the set covered by* $\mathbf{C}$ *implies every* $R \in K$ *critically covers a square in z. If not, replace R by* R_1 *or* R_2 *(see Step* 2).

3) Define the sets of edge squares shown in Fig. 28,

$$R_1=(r_2,q_1),\qquad R_2=(q_2,r_1),$$
$$Q_1=[q_1,s_1)\qquad Q_2=(s_2,q_2].$$

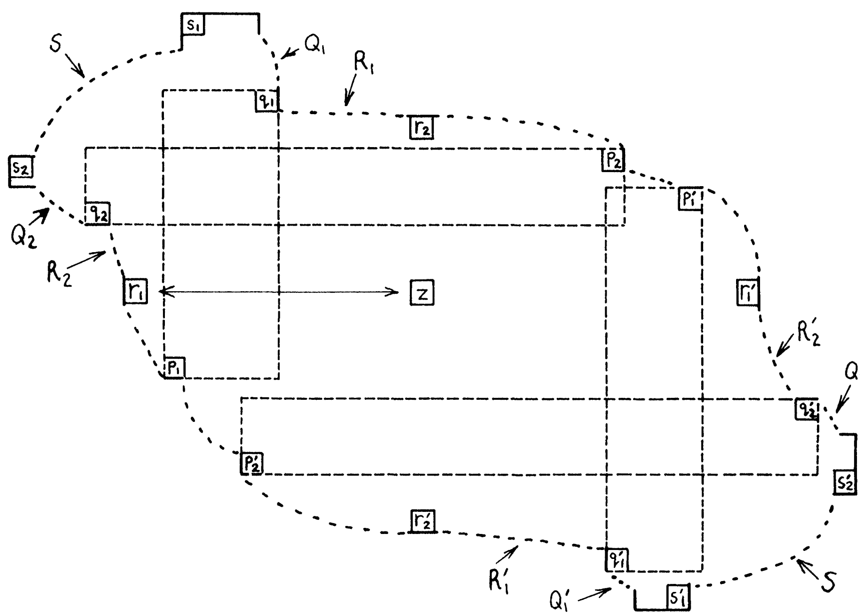


FIG. 28.

We claim that every rectangle in $\mathbf{K}_i$ critically covers a square in $S\cup Q_i$. By 2), $R\in\mathbf{K}_i$ critically covers a square in $S\cup Q_i\cup R_i$. If it covered a square in R_i, however, this would violate the assumption that $\Box p_1q_1$ is a rectangle in $\mathbf{K}_i$ closest to z.

4). If $R\in\mathbf{C}$ covers a square in R_i, then it must critically cover a square in $R_{i'}\cup Q_{i'}\cup S$, where $i'=3-i$. Such $R\notin\mathbf{K}$. Otherwise, we can give a proof similar to that in 2) to contradict the maximality of $\mathbf{C}$ or the choice of $\Box p_iq_i$.

We conclude from 1)–4) that every rectangle $R\in\mathbf{C}$ satisfies exactly one of two possibilities and define $a:\mathbf{C}\to\mathcal{B}$ thereby:

i. R critically covers a square in $S\cup Q_1\cup Q_2$ or $S\cup Q_1'\cup Q_2'$. If it critically covers some $s\in S$, set $a(R)=s$, otherwise set $a(R)$ to any square in $\bigcup Q$ critically covered by R.

ii. If i does not hold, then R matches a square $a=a(R)\in R_1$ with a square in R_2, or $a=a(R)\in R_1'$ with a square in R_2'. This follows from 4. (Note the asymmetry. $a(R)$ is always chosen from R_1 or R_1'.)

Let $\mathcal{A}=\{a(R)|R\in\mathbf{C}\}$. $a(R)\in\mathcal{A}$ is always critically covered by R so $|\mathcal{A}|=|\mathbf{C}|$. As defined, $\mathcal{A}\subseteq\bar{z}$, so as long as $\mathcal{A}$ is an antirectangle $\mathcal{A}\cup\{z\}$ is the desired maximum antirectangle. Clearly, S, $Q_i\cup R_i$, and $Q_i'\cup R_i'$ are each antirectangles (consider corner types). We conclude the proof that $\mathcal{A}$ is an antirectangle by showing each of the three remaining possible ways for two squares in a common rectangle to be both in $\mathcal{A}$ leads to contradiction.

5) Suppose $\{s,q\}\subseteq\mathcal{A}$ with $s\in S$, $q\in(\bigcup Q)\cup(\bigcup R)$, and $(s,q)\in G$. This cannot happen because (by 1) s is a corner square and is contained in a unique maximal rectangle, so q cannot be critically covered.

6) Suppose say, $q \in Q_1, q' \in Q_2$ and $(q, q') \in G$. The board structure implies $(q_1, q_2) \in G$. $\mathbf{C}\backslash\{\Box p_1q_1, \Box p_2q_2\} \cup \{\Box p_1p_2, \Box q_1q_2\}$ covers more than $\mathbf{C}$, including z. This contradicts the maximality of $\mathbf{C}$.

7) Suppose say, $\{r, q\} \subseteq \mathcal{A}$ with $r \in R_1$, $q \in Q_2$, and $(r, q) \in G$. Then $(r, q_2) \in G$. $r \in \mathcal{A}$ implies $r = a(R)$ for some $R = \Box rr'$ that satisfies ii; that is, R matches $r' \in R_2$ with r. We do another switch, $\mathbf{C}\backslash\{\Box rr', \Box q_2p_2\} \cup \{\Box r'p_2, \Box rq_2\}$. Again, the maximality of the set covered by $\mathbf{C}$ is contradicted.

REFERENCES

[1] C. BERGE, *Perfect graphs*, in Studies in Graph Theory Part 1, D. R. Fulkerson, ed., Studies in Mathematics 11, Mathematical Association of America, 1975, pp. 1–22.

[2] R. BRUALDI, *Transversal theory and graphs*, in Studies in Graph Theory, Part 1, D. R. Fulkerson, ed., Studies in Mathematics 11, Mathematical Association of America, 1975, pp. 23–88.

[3] F. R. K. CHUNG, Personal communication, February, 1979.

[4] C. C. LIU, *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, 1968.

[5] W. MASEK, Personal communication, June, 1979.

[6] D. R. WOODALL, *Minimax theorems in graph theory*, in Selected Topics in Graph Theory, L. W. Beineke and R. J. Wilson, eds., Academic Press, New York, 1978, pp. 237–269.