

This homework is counted 15% of the whole course. There are total 80 marks.

You are allowed to discuss with others and use any references, but if you do so please list your collaborators and cite your references for each question. This will not affect your marks. In any case, you must write your own solutions (meaning that after you understand the solutions either by your own or from discussions with others or from references, you must write the solutions in your own words without any help from others and without any help from the references). Not writing your own solutions or not listing your collaborators or not citing your references may be considered plagiarism.

1. Bipartite Graphs

- (a) (10 marks) Consider the adjacency matrix A of an undirected connected graph G . Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A . Prove that $\lambda_1 = -\lambda_n$ if and only if G is bipartite.
(Hint: you may use the corollary of the Perron-Frobenius theorem stated in the notes.)
(Remark: you could get half the marks if you assume that G is regular, or if you prove the same statement for the normalized adjacent matrix $\mathcal{A} = D^{-1/2}AD^{-1/2}$.)

2. Cycles

- (a) (10 marks) Compute the Laplacian spectrum of C_n (the cycle with n vertices).

3. Hypercubes

A hypercube of n -dimension is an undirected graph with 2^n vertices. Each vertex corresponds to a string of n bits. Two vertices have an edge if and only if their corresponding strings differ by exactly one bit.

- (a) (10 marks) Given two undirected graphs $G = (V, E)$ and $H = (U, F)$, we define $G \times H$ as the undirected graph with vertex set $V \times U$ and two vertices $(v_1, u_1), (v_2, u_2)$ have an edge if and only if either (1) $v_1 = v_2$ and $u_1u_2 \in F$ or (2) $u_1 = u_2$ and $v_1v_2 \in E$. Let x be an eigenvector of the Laplacian of G with eigenvalue α , and let y be an eigenvector of the Laplacian of H with eigenvalue β . Prove that we can use x and y to construct an eigenvector of the Laplacian of $G \times H$ with eigenvalue $\alpha + \beta$.
- (b) (10 marks) Use (a) to compute the spectrum of the hypercube of n dimension.
- (c) (bonus 10 marks) Show that the spectral algorithm for conductance may return a set S of conductance $\Omega(\sqrt{\phi(S^*)})$ in a hypercube, where S^* is the set with minimum conductance in the hypercube and $\phi(S^*)$ is the conductance of S^* .

4. Line Graphs

Let $G = (V, E)$ be an undirected d -regular graph. The line graph H of G has vertex set E and two vertices e_1, e_2 are adjacent if and only if their corresponding edges in G have a common vertex.

- (a) (10 marks) Let $V = \{1, \dots, n\}$, $e = ij$, and B_e be the n -dimensional vector with $+1$ in the i -th entry and $+1$ in the j -th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G . Prove that the adjacency matrix of H is $B^T B - 2I$. Conclude that the smallest eigenvalue of the adjacency matrix of H is at least -2 .
- (b) (10 marks) Use (a) to compute the spectrum of the adjacency matrix of H in term of the spectrum of the adjacency matrix of G .
(Hint: (1) Write the adjacency matrix of G in terms of B , (2) Show that $\det(I - XY) = \det(I - YX)$ for any X, Y with appropriate dimensions (not necessary square matrices).)

5. Spanning Trees

Let $G = (V, E)$ be an undirected graph.

- (a) (10 marks) Let $V = \{1, \dots, n\}$, $e = ij$, and B_e be the n -dimensional vector with $+1$ in the i -th entry and -1 in the j -th entry and 0 otherwise. Let B be an $n \times m$ matrix where the columns are b_e and m is the number of edges in G . Prove that the determinant of any $(n - 1) \times (n - 1)$ submatrix of B is ± 1 if and only if the $n - 1$ edges corresponding to the columns form a spanning tree of G .
- (b) (10 marks) Let L be the Laplacian matrix of G and let L' be the matrix obtained from L by deleting the last row and last column. Use (a) to prove that $\det(L')$ is equal to the number of spanning trees in G .
(Hint: Look up the Cauchy-Binet formula in wikipedia.)