CSC2110 Discrete Mathematics
Supplementary exercise 1 solutions

---

1 Basics

1. The truth table is as follow

p	q	r	p ⊕ q ⊕ r
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	F
T	T	T	T

2. $(\neg p \land \neg q \land r) \lor (\neg p \land q \land \neg r) \lor (p \land \neg q \land \neg r) \lor (p \land \neg q \land r) \lor (p \land q \land \neg r)$
$\equiv((\neg p \lor p) \land \neg q \land r) \lor ((\neg p \lor p) \land q \land \neg r) \lor (p \land \neg q \land \neg r)$
$\equiv(\neg q \land r) \lor (q \land \neg r) \lor (p \land \neg q \land \neg r)$
$\equiv q \oplus r \lor (p \land \neg q \land \neg r)$

3. $\neg(\neg q \land \neg(\neg q \lor s)) \lor (q \land (r \to r))$
$\equiv\neg(\neg q \land \neg(\neg q \lor s)) \lor q$
$\equiv(q \lor (\neg q \lor s)) \lor q$
$\equiv(q \lor \neg q) \lor s \lor q$
$\equiv$true

4. Only (c) must be true.

5. Only (c) must be true.

6. (a) $\forall x(P(x) \to Q(x))$
   (b) $\exists x(P(x) \land Q(x))$
   (c) $\forall x(Q(x) \to P(x))$
   (d) $\exists x(P(x) \land Q(x))$

7. We prove this by case:

Case 1 $x \geq y$: $\max(x, y) = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$

Case 2 $x < y$: $\max(x, y) = \frac{x+y-(x-y)}{2} = \frac{2y}{2} = y$

8. Assume $a + br$ is rational, i.e. $a + br = \frac{m}{n}$ for some integer m and n. Then $r = (\frac{m}{n} - a)/b = \frac{m-na}{nb}$, which means r is also rational. Here we get a contradiction and hence $a + br$ must be irrational.

9. Among the 4 glasses you flip, no matter none, 1, 2, 3 or 4 of them are upside down, the changes on the number of upside down glasses must be even. In the other word, since the number of upside down glassed is odd in the beginning, it will remain odd all the time. As a result, it is not possible to change all the glasses right-side-up.

2 Medium

1. Let $x = i+f$ where i is the integral part of x and f is the fractional part of x, where $0 \leq f < 1$. The case $f = 0$ is obvious so here we consider $0 < f < 1$. We get $\lceil x \rceil = \lceil i + f \rceil = i + 1$ and $-\lfloor -x \rfloor = -\lfloor -i - f \rfloor = -(-i - 1) = i + 1$. Hence $\lceil x \rceil = -\lfloor -x \rfloor$.

2. Let G, C, B and H be the proposition that the butler, cook, gardener and handyman is telling the truth respectively. Then from the problem description we get

   (i) $B \to C$
   (ii) $\neg(C \wedge G)$
   (iii) $G \vee H$
   (iv) $H \to \neg C$

   Now suppose C is true, then by (iv) we get H is false, then by (iii) we get G is true. By (ii) we get C is false, which is a contradiction. As a result C must be false. Furthermore, B must be false by (i). Here we cannot proceed any further, so we know the cook and butler must be lying.

3. Suppose $\sqrt{6}$ is rational, so we can write $\sqrt{6} = \frac{m}{n}$ so that m and n do not have common factor other than 1. So we get

$$\begin{aligned} 6 &= \frac{m^2}{n^2} \\ 6n^2 &= m^2 \end{aligned}$$

   Since left hand side is even, m must be even, put $n = 2k$. Then

$$\begin{aligned} 6n^2 &= (2k)^2 \\ 3n^2 &= 2k^2 \end{aligned}$$

   Now right hand side is even, n must also be even. This contradict the assumption that n and m do not have common factor other than 1. Hence $\sqrt{6}$ is irrational.

4. When $n = 4$, $2^4 = 16 = 4^2$. Assume $2^k \geq k^2$, now consider $n = k + 1$. We get $2^{k+1} = 2(2^k) \geq 2k^2 \geq (k+1)^2$. Note that $2k^2 - (k+1)^2 = k^2 - 2k - 1 > 0$ because $k^2 - 2k - 1$ can be proved to be positive $\forall k \geq 4$. Hence the proposition is true $\forall n \geq 4$.

5. (a) Write $ak = b$ for some k, then $b^2 = (ak)^2 = a^2k^2$. So $a^2 \mid b^2$.
   (b) Assume $a^2 - 2$ is divisible by 4, then $a^2 - 2 = 4k$ for some k. Observe a must be even, so substitute $a = 2b$. We get $4b^2 = 4k + 2$. Note that left side is divisible by 4 while right side is not, which is a contradiction. Thus $a^2 - 2$ is not divisible by 4.

6. Assume there exists integers a, b and c such that a and b are odd and $a^2 + b^2 = c^2$. Since a and b are odd we write them as $a = 2p + 1$ and $b = 2q + 1$. We get

$$(2p+1)^2 + (2q+1)^2 = c^2$$
$$4p^2 + 4p + 4q^2 + 4q + 2 = c^2$$

Since right side is even, c must be even and we substitute $c = 2k$. Then we get

$$4p^2 + 4p + 4q^2 + 4q + 2 = 4c^2$$

which leads to contradiction because right side is divisible by 4 while left side does not. Hence the result.

7. First of all, $c_3 = c_4 = c_5 = 6$, so the hypothesis is true for $n = 3$, 4 and 5. Now assume the hypothesis is true for $n = 3, 4, \cdots, k$, where $k \geq 5$. Consider $n = k + 1$:

$$\begin{aligned} c_{k+1} &= c_{\lfloor (k+1)/2 \rfloor} + 3 & \\ &\leq 2 \lfloor (k+1)/2 \rfloor + 3 & \text{by assumption} \\ &\leq (k+1) + 3 & \\ &\leq 2(k+1) & \text{since } k \geq 5 \end{aligned}$$

Hence by induction the hypothesis is true for all $n \geq 3$.

8. *Existence*
Let S be set of non-negative integers of the form $n - dk$. It is easy to show that S is non-empty. By well-ordering principle S contains a least element r. Let the corresponding value of k be q. Then we get $n - dq = r$, which implies $n = dq + r$. Now we have to show $r < d$. Suppose $r \geq d$, then there exists $r' = n - d(q+1) = n - dq - d = r - d \geq 0$, which is smaller than r and belongs to S. Here comes a contradiction because r is the smallest element in S. Hence $r < d$.

   Uniqueness
Assume there exists q_1, q_2, r_1, r_2 such that $r_1 \neq r_2$ and $n = dq_1 + r_1 = dq_2 + r_2$. Then we get $d(q_1 - q_2) = r_2 - r_1$. Without loss of generality assume $q_2 \leq q_1$. So we get $r_2 > r_1$. Since $r_2 < d \leq d + r_1$ which implies $r_2 - r_1 < d$. As a result, we have $d \mid r_2 - r_1$ and $r_2 - r_1 < d$, and we can conclude $r_2 - r_1 = 0$. Hence a contradiction and we conclude r must be unique.

9. Number of out-of-order pairs mod 2 remains unchanged regardless what move you make. The proof for this is very similar to the one for 15-puzzle.