CSC2110 Discrete Mathematics
Supplementry Exercise 1 **Last Updated: 10-Oct-2009**

1 Basics

1. Write a truth table for $p \oplus q \oplus r$.

2. Write a logical formula for the below truth table and then simplify it. Use the approach mentioned in lecture 1 slide 46.

p	q	r	Output
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	F

3. Simplify the following statement. $\neg(\neg q \wedge \neg(\neg q \vee s)) \vee (q \wedge (r \rightarrow r))$

4. Assume that $\exists x \forall y P(x, y)$ is true and the domain of them is nonempty. Which of the following must be true?
   (a) $\forall x \forall y P(x, y)$
   (b) $\forall x \exists y P(x, y)$
   (c) $\exists x \exists y P(x, y)$

5. Assume that $\forall x \exists y P(x, y)$ is false and the domain of them is nonempty. Which of the following must be false?
   (a) $\forall x \forall y P(x, y)$
   (b) $\exists x \forall y P(x, y)$
   (c) $\exists x \exists y P(x, y)$

6. Let $P(x)$ denote the statement "x is an accountant" and let $Q(x)$ denote the statement "x owns a Porsche". Write each statement below in first order logic.
   (a) All accountants own Porsches.
   (b) Some accountant owns a Porsche.
   (c) All owners of Porsches are accountants
   (d) Someone who owns a Porsche is an accountant.

7. Prove that for all real numbers x and y the following is true

$$\max(x, y) = \frac{x + y + \mid x - y \mid}{2}$$

8. If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irrational.

9. There are 7 glasses on a table, all standing upside down. You are allowed to turn over any 4 of them in one move. Is it possible to have all the glasses right-side-up?

2 Medium

1. Prove $\lceil x \rceil = - \lfloor -x \rfloor$ for any real number x.

2. A detective has interviewed four witnesses to a crime. From their stories, the detective has concluded that:

   (a) If the butler is telling the truth, then so is the cook.

   (b) The cook and the gardener cannot both be telling the truth.

   (c) The gardener and the handyman are not both lying.

   (d) If the handyman is telling the truth then the cook is lying.

   Who must be lying? There may be more than one liar. Show your steps.

3. Prove that $\sqrt{6}$ is irrational.

4. Show that $2^n \geq n^2$, $n = 4, 5, ...$

5. Prove the followings are true.

   (a) For all integers a and b, if $a \mid b$ then $a^2 \mid b^2$.

   (b) For any integer a, $(a^2 - 2)$ is not divisible by 4.

6. Prove that if a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b must be even.

7. Define the sequence $c_0, c_1, \cdots$ by the equations $c_0 = 0$ and $c_n = c_{\lfloor n/2 \rfloor} + 3$ for all $n > 0$. Prove that $c_n \leq 2n$ for all $n \geq 3$.

8. Prove that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

9. Give an invariant for the 5×5 version of 15-puzzle (aka 24-puzzle).

3 Hard

1. Let $n \geq 2$ be a natural number. We consider the following game. Two players write a sequence of 0s and 1s. They start with an empty line and alternate their moves. In each move, a player writes 0 or 1 to the end of the current sequence. A player loses if his digit completes a block of n consecutive digits that repeats itself in the sequence for the second time (the two occurrences of the block may overlap). For instance, for $n = 4$, a sequence produced by such a game may look as follows: 00100001101011110011(the second player lost by the last move because 0011 is repeated).

   (a) Prove that the game always finishes after finitely many steps.

   (b) Suppose that n is odd. Prove that the second player (the one who makes the second move) has a winning strategy.

2. Lay a strip of paper left-to-right in front of you. Now fold it by taking the right end to the left end. Press flat so it is folded in half. Unfold the paper strip by reversing the folding process. With the paper laid on the table observe that some the crease point up(U) and some point down(D).

   (a) How many creases are there after n folds?

   (b) For any number $n > 0$ folds what invariant relationship exists between the number of U crease and the number of D creases.

   (c) If you are given the sequence of U and D for $n > 0$ folds, how can you determine the sequence for $n + 1$ folds?

   (d) Give an inductive argument that if each crease is set to a 90 degree angle then the paper will never cross itself.

3. It is known that n straight lines in the plane divide the plane into $(n^2 + n + 2)/2$ regions. Assume that no two lines are parallel and that no three lines have a common point. Now, show that the regions can be colored red and green so that no two regions that share an edge are the same color.

4. Suppse $n > 1$ people are positioned in a field, so that each has a unique nearest neighbour. Suppse further that each person has a pie that is hurled at the nearest neighbour. A survivor is a person that is not hit by a pie. Use induction on n to show that if n is odd, there is always at least one survivor.