Propositional Logic

1. **T/F**
   (a) F.
   (b) T.
   (c) F.
   (d) T.
   (e) T.
   (f) F.
   (g) T.
2. **SQ**
   (a) $h \wedge c \wedge \neg o$.
   (b) $c \vee h$.
   (c) $(c \vee h) \wedge \neg(c \wedge h)$.
   (d) $\neg(h \vee c \vee o)$.
   (e) $\neg(h \wedge c) \wedge o$.
   (f) $o \wedge \neg(c \vee h)$.
3. **SQ**
   (a) $(p \wedge \neg q) \vee (\neg p \wedge q)$.
   (b) $\neg p \wedge \neg q$.
   (c) $\neg p \wedge q$.
   (d) $p \wedge \neg q$.
4. **SQ**
   (a)

$$
\begin{aligned}
f(p,q,r) =& (p \wedge q \wedge r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r) \\
=& (p \wedge q \wedge r) \vee ((\neg p \wedge \neg q) \wedge (r \vee \neg r)), && \text{Distributive Law} \\
=& (p \wedge q \wedge r) \vee (\neg p \wedge \neg q)
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(p,q,r) =& \neg(p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge q \wedge \neg r) \wedge \neg(\neg p \wedge \neg q \wedge \neg r) \\
=& (\neg p \vee \neg q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee r), && \text{De Morgan's Law} \\
=& ((\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge (p \vee q)) \vee r, && \text{Distributive Law} \\
=& (((\neg p \wedge p) \vee \neg q) \wedge (p \vee q)) \vee r, && \text{Distributive Law} \\
=& (\neg q \wedge (p \vee q)) \vee r \\
=& (\neg q \wedge p) \vee (\neg q \wedge q) \vee r, && \text{Distributive Law} \\
=& (\neg q \wedge p) \vee r
\end{aligned}
$$

5. **SQ**

(a) They are not equivalent. E.g. $p = F, q = T, r = T$.

(b) They are equivalent.

p	q	$\neg(p \oplus q)$	$p \leftrightarrow q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

(c) They are equivalent. $\neg(p \vee q \vee r) = \neg(p \vee (q \vee r)) = \neg p \wedge \neg(p \vee r) = \neg p \wedge \neg p \wedge \neg r$.

(d) They are equivalent. By absorption laws: $p \wedge (p \vee q) = p = p \vee (p \wedge q)$.

(e) They are not equivalent. E.g. $p = F, q = T, r = F$.

6. **SQ**

(a)

$$
\begin{aligned}
& \neg((\neg p \wedge (\neg q \vee p)) \vee \neg r) \\
=& \neg(((\neg p \wedge \neg q) \vee (\neg p \wedge p)) \vee \neg r), && \text{Distributive Law} \\
=& \neg((\neg p \wedge \neg q) \vee \neg r) \\
=& \neg(\neg p \wedge \neg q) \wedge r, && \text{De Morgan's Law} \\
=& (p \vee q) \wedge r, && \text{De Morgan's Law}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& (\neg p \wedge \neg(p \wedge r)) \vee ((q \vee (q \wedge r)) \wedge (q \vee s)) \\
=& (\neg p \wedge \neg(p \wedge r)) \vee (q \wedge (q \vee s)), && \text{Absorption Law} \\
=& (\neg p \wedge \neg(p \wedge r)) \vee q, && \text{Absorption Law} \\
=& (\neg p \wedge (\neg p \vee \neg r)) \vee q, && \text{De Morgan's Law} \\
=& \neg p \vee q, && \text{Absorption Law}
\end{aligned}
$$

(c)

$$
\begin{aligned}
&((\neg p \wedge q) \wedge (q \wedge r)) \wedge \neg q \\
=&(\neg p \wedge (q \wedge q) \wedge r) \wedge \neg q \\
=&(\neg p \wedge q \wedge r) \wedge \neg q \\
=&\neg p \wedge (q \wedge \neg q) \wedge r \\
=&\neg p \wedge r \wedge F \\
=&F
\end{aligned}
$$

(d)

$$
\begin{aligned}
&\neg(\neg q \wedge \neg(\neg q \vee s)) \vee (q \wedge (r \to r)) \\
=&\neg(\neg q \wedge (q \wedge \neg s)) \vee (q \wedge T), && \text{De Morgan's Law} \\
=&\neg(F \wedge \neg s) \vee q \\
=&T \vee q \\
=&T
\end{aligned}
$$

(e)

$$
\begin{aligned}
&\neg p \wedge (p \vee q) \vee ((q \vee (p \wedge p)) \wedge (p \vee \neg q)) \\
=&\neg p \wedge (p \vee q) \vee ((q \vee p) \wedge (p \vee \neg q)) \\
=&(\neg p \wedge p) \vee (\neg p \wedge q) \vee (p \vee (q \wedge \neg q)), && \text{Distributive Law} \\
=&(\neg p \wedge q) \vee p \\
=&(p \vee \neg p) \wedge (p \vee q), && \text{Distributive Law} \\
=&p \vee q
\end{aligned}
$$

7. **SQ**

   (a) $\neg p \vee q$.

   (b) $\neg p \vee q$.

   (c) $p \vee \neg q$.

   (d) $p \vee \neg q$.

   (e) $p \wedge \neg q$.

8. **SQ**

   (a)

p	q	$p \to q$	$q \to p$	$p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	F

   The last row show that this argument is invalid.

(b)

p	q	$p \to q$	q	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

The third row show that this argument is invalid.

(c)

p	q	r	p	$p \to q$	$\neg q \to r$	r
T	T	T	T	T	T	T
T	T	F	T	T	F	F
T	F	T	T	F	T	T
T	F	F	T	F	T	F
F	T	T	F	T	T	T
F	T	F	F	T	F	F
F	F	T	F	T	T	T
F	F	F	F	T	T	F

This argument is valid.

(d)

p	q	r	$p \wedge q \to \neg r$	$p \vee \neg q$	$\neg q \to p$	$\neg r$
T	T	T	F	T	T	F
T	T	F	T	T	T	T
T	F	T	T	T	T	F
T	F	F	T	T	T	T
F	T	T	T	F	T	F
F	T	F	T	F	T	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

The third row show that this argument is invalid.

(e)

p	q	r	$p \to q \vee r$	$\neg q \vee \neg r$	$\neg p \vee \neg r$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	T	F
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

The third row show that this argument is invalid.

(f)

p	q	$p \to q$	$\neg p$	$\neg q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

The third row show that this argument is invalid.

(g)

p	q	r	$p \to q \vee \neg r$	$q \to p \wedge r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	T	T	F
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

The forth row show that this argument is invalid.

9. **SQ**

(a)

$$\begin{array}{ll} & \neg w \\ & u \vee w \\ \therefore & u \end{array} \qquad \begin{array}{ll} & u \\ & u \to \neg p \\ \therefore & \neg p \end{array} \qquad \begin{array}{ll} & \neg p \\ & \neg p \to r \wedge \neg s \\ \therefore & r \wedge \neg s \end{array} \qquad \begin{array}{ll} & r \wedge \neg s \\ \therefore & \neg s \end{array} \qquad \begin{array}{ll} & \neg s \\ & t \to s \\ \therefore & \neg t \end{array}$$

(b)

$$\begin{array}{ll} & \neg t \\ & p \to t \\ \therefore & \neg p \end{array} \qquad \begin{array}{ll} & \neg p \\ \therefore & \neg p \vee q \end{array} \qquad \begin{array}{ll} & \neg p \vee q \\ & \neg p \vee q \to r \\ \therefore & r \end{array} \qquad \begin{array}{ll} & \neg p \\ & r \\ \therefore & \neg p \wedge r \end{array} \qquad \begin{array}{ll} & \neg p \wedge r \\ & \neg p \wedge r \to \neg s \\ \therefore & \neg s \end{array} \qquad \begin{array}{ll} & \neg s \\ & s \vee \neg q \\ \therefore & \neg q \end{array}$$

(c)

$$\begin{array}{ll} & \neg r \\ & q \to r \\ \therefore & \neg q \end{array} \qquad \begin{array}{ll} & \neg q \\ & p \vee q \\ \therefore & p \end{array} \qquad \begin{array}{ll} & \neg q \\ & \neg q \to u \wedge s \\ \therefore & u \wedge s \end{array} \qquad \begin{array}{ll} & u \wedge s \\ \therefore & s \end{array} \qquad \begin{array}{ll} & p \\ & s \\ \therefore & p \wedge s \end{array} \qquad \begin{array}{ll} & p \wedge s \\ & p \wedge s \to t \\ \therefore & t \end{array}$$

(d)

$$\begin{array}{ll} & \neg s \\ & \neg s \to \neg t \\ \therefore & \neg t \end{array} \qquad \begin{array}{ll} & \neg t \\ & w \vee t \\ \therefore & w \end{array} \qquad \begin{array}{ll} & \neg s \\ & \neg q \vee s \\ \therefore & \neg q \end{array} \qquad \begin{array}{ll} & \neg q \\ & p \to q \\ \therefore & \neg p \end{array} \qquad \begin{array}{ll} & \neg s \\ & r \vee s \\ \therefore & r \end{array} \qquad \begin{array}{ll} & r \\ & \neg p \\ \therefore & \neg p \wedge r \end{array} \qquad \begin{array}{ll} & \neg p \wedge r \\ & \neg p \wedge r \to u \\ \therefore & u \end{array}$$

$$\begin{array}{ll} & u \\ & w \\ \therefore & u \wedge w \end{array}$$

10. **T/F**

(a) T. A contradiction in hypothesis makes an argument valid.
Definition of valid argument: hypothesis 1 $\wedge$ hypothesis 2 $\wedge$ ... $\to$ conclusion

(b) F. When p = F, we have true hypothesis but false conclusion.

(c) T. A tautology makes an argument valid.

(d) F. Possible to have a dry floor (indoor) in a rainy day.

(e) T.

(f) T.

11. **SQ**
Let b be the butler is telling the truth.
c be the cook is telling the truth.

g be the gardener is telling the truth.
h be the handyman is telling the truth.
So we have

(a) $b \to c$
(b) $\neg(c \wedge g) = \neg c \vee \neg g$
(c) $\neg(\neg g \wedge \neg h) = g \vee h$
(d) $h \to \neg c$

Assume $h = F$:

$$\begin{array}{llll} & \neg h & & g \\ & g \vee h & & \neg c \vee \neg g \\ \therefore & g & \therefore & \neg c \end{array}$$

Assume $h = T$:

$$\begin{array}{ll} & h \\ & h \to \neg c \\ \therefore & \neg c \end{array}$$

Therefore the cook must be is lying.

$$\begin{array}{ll} & \neg c \\ & b \to c \\ \therefore & \neg b \end{array}$$

As a result, the butler and cook must be lying, and we cannot determine whether the gardener and the handyman is lying.

First Order Logic

1. **SQ**

   (a) Let $F(x)$ be "Disk x has more than 10 kilobytes of free space" and $S(x)$ be "Mail message x can be saved".
   $(\exists x, F(x)) \to (\exists x, S(x))$

   (b) Let $A(x)$ be "Alert x is active", $Q(x)$ be "Message x is queued" and $T(x)$ be "Message x is transmitted".
   $(\exists x, A(x)) \to (\forall x, Q(x) \to T(x))$

   (c) Let $T(x, y)$ be "x has tried food y" and $L(x, y)$ be "x likes food y".
   $(\forall x, (\forall y, T(x, y) \to (\exists y, \neg L(x, y))$

   (d) Let $F(x)$ be "x is a friend of the host", $C(x)$ be "x is charged".
   $\forall x, \neg F(x) \to C(x)$.

2. **SQ**

   (a) $\exists r, (O(r) \wedge B(r)) \rightarrow \exists p, L(p)$.
   If there are some out of order and busy routers, then some packets are lost.

   (Negation) $\exists r, (O(r) \wedge B(r)) \wedge \forall p, \neg L(p)$.
   There are some out of order and busy routers and no packet is lost.

   (b) $\forall r, B(r) \rightarrow \exists p, Q(p)$.
   If all routers are busy, then some packets are queued.

   (Negation) $\forall r, B(r) \wedge \forall p, \neg Q(p)$.
   All routers are busy and no packets are queued.

   (c) $\exists p, (Q(p) \wedge L(p)) \rightarrow \exists r, O(r)$.
   If some packets are queued and lost, then there are some out of order routers.

   (Negation) $\exists p, (Q(p) \wedge L(p)) \wedge \forall r, \neg O(r)$.
   Some packets are queued and lost but all routers are working normally.

   (d) $(\exists r, B(r) \wedge \forall p, Q(p)) \rightarrow \exists p, L(p)$.
   If some routers are busy and all packets are queued, then some packets are lost.

   (Negation) $(\exists r, B(r) \wedge \forall p, Q(p)) \wedge \forall p, \neg L(p)$.
   Some routers are busy and all packets are queued but no packet is lost.

3. **SQ**

   (a) $\forall x \in S, \exists y \in C, T(x, y)$.
   Every students takes some courses.

   (Negation) $\exists x \in S, \forall y \in C, \neg T(x, y)$.
   Some student doesn't take any course.

   (b) $\forall (x, y) \in S \times C, L(x, y)$.
   All students like all the courses.

   (Negation) $\exists (x, y) \in S \times C, \neg L(x, y)$.
   Some student doesn't like some courses.

   (c) $\exists x \in S, \forall y \in C, (L(x, y) \rightarrow T(x, y))$.
   For some students, if they like any courses, they will take all of them.

(Negation) $\forall x \in S, \exists y \in C, (L(x, y) \wedge \neg T(x, y))$.
Every student has some courses that they like but didn't take it.

(d) $\exists (x, y) \in S \times C, \neg L(x, y) \leftrightarrow \neg T(x, y) \equiv \exists (x, y) \in S \times C, L(x, y) \leftrightarrow T(x, y)$.
For some students, they take some courses if and only if they like the courses.

(Negation) $\forall (x, y) \in S \times C, (\neg L(x, y) \leftrightarrow T(x, y))$.
For all students and courses, a students take a course if and only if they dislike it.

4. **SQ**

(a)
$$\begin{array}{ll} & \forall x \in S, T(x) \\ \therefore & T(a) \end{array} \qquad \begin{array}{ll} & T(a) \\ & \neg P(a) \vee \neg T(a) \\ \therefore & \neg P(a) \end{array} \qquad \begin{array}{ll} & \neg P(a) \\ & \exists x \in S, Q(x) \rightarrow P(x) \\ \therefore & \neg Q(x) \end{array} \qquad \begin{array}{ll} & \neg Q(a) \\ & \forall x \in S, \neg Q(x) \rightarrow R(x) \\ \therefore & R(a) \end{array}$$

(b)
$$\begin{array}{ll} & (\forall x, T(x)) \wedge (\forall x, \neg R(x)) \\ \therefore & \forall x, \neg R(x) \end{array} \qquad \begin{array}{ll} & \forall x, \neg R(x) \\ \therefore & \neg R(a) \end{array} \qquad \begin{array}{ll} & \neg R(a) \\ & \forall x, \neg R(x) \rightarrow S(x) \\ \therefore & S(a) \end{array}$$

$$\begin{array}{ll} & S(a) \\ & \neg S(a) \vee \neg U(a) \vee V(a) \\ \therefore & \neg U(a) \vee V(a) \end{array} \qquad \begin{array}{ll} & \neg U(a) \vee V(a) \\ & \forall x, V(x) \rightarrow W(x) \\ & \forall x, \neg W(x) \rightarrow U(x) \\ \therefore & W(a) \end{array}$$

5. **T/F**

(a) T.

(b) T.

(c) F.

(d) F.

(e) T.

(f) F. $\neg \forall y, \forall x, P(x, y) \equiv \exists y, \neg(\forall x, P(x, y)) \equiv \exists y, \exists x, \neg P(x, y) \not\equiv \forall x, \exists y, \neg P(x, y) \equiv \forall x, \neg \forall y, P(x, y)$.

(g) T.

(h) F. When $x = 1$, no such y exists.

(i) F. It has the meaning of all integer p is prime. The orginal statement maybe written as $\forall p \in \mathbb{Z}, \exists q \in \mathbb{Z}, (\text{prime}(q) \wedge (q > p))$.

Direct Proofs

1. Suppose $a|b$.

By definition of divisibility, this means $b = ac$ for some integer c.

Squaring both sides of this equation produces $b^2 = a^2c^2$.

Then $b^2 = a^2d$, where $d = c^2 \in \mathcal{Z}$.

By definition of divisibility, this means $a^2|b^2$.

2. Suppose $7|4a$.

   By definition of divisibility, this means $4a = 7c$ for some integer c.

   Since $4a = 2(2a)$ it follows that $4a$ is even, and since $4a = 7c$, we know $7c$ is even.

   But then c cant be odd, because that would make $7c$ odd, not even.

   Thus c is even, so $c = 2d$ for some integer d.

   Now go back to the equation $4a = 7c$ and plug in $c = 2d$. We get $4a = 14d$.

   Dividing both sides by 2 gives $2a|7d$.

   Now, since $2a|7d$, it follows that $7d$ is even, and thus d cannot be odd.

   Then d is even, so $d = 2e$ for some integer e.

   Plugging $d = 2e$ back into $2a = 7d$ gives $2a = 14e$.

   Dividing both sides of $2a = 14e$ by 2 produces $a = 7e$.

   Finally, the equation $a = 7e$ means that $7|a$, by definition of divisibility.

3. From the formula $\binom{p}{k} = \frac{p!}{(p-k)!k!}$, we get

$$p! = \binom{p}{k}(p-k)!k!.$$

   Now, since the prime number p is a factor of $p!$ on the left, it must also be a factor of $\binom{p}{k}(p-k)!k!$ on the right. Thus the prime number p appears in the prime factorization of $\binom{p}{k}(p-k)!k!$.

   Now, $k!$ is a product of numbers smaller than p, so its prime factorization contains no ps. Similarly the prime factorization of $(p-k)!$ coSntains no ps. But we noted that the prime factorization of $\binom{p}{k}(p-k)!k!$ must contain a p, so it follows that the prime factorization of $\binom{p}{k}$ contains a p. Thus $\binom{p}{k}$ is a multiple of p, so p divides $\binom{p}{k}$.

4. By definition, $\binom{2n}{n}$ is the number of n-element subsets of a set A with $2n$ elements. For each subset $X \subset A$ with $|X| = n$, the complement $\overline{X}$ is a different set, but it also has $2n - n = n$ elements. Imagine listing out all the n-elements subset of a set A. It could be done in such a way that the list has form

$$X_1, \overline{X_1}, X_2, \overline{X_2}, X_3, \overline{X_3}, X_4, \overline{X_4}, X_5, \overline{X_5}, \ldots$$

   This list has an even number of items, for they are grouped in pairs. Thus $\binom{2n}{n}$ is even.

5. Assume $a, b, c \in \mathcal{N}$ with $c \leq b \leq a$. Then we have $\binom{a}{b}\binom{b}{c} = \frac{a!}{(a-b)!b!}\frac{b!}{(b-c)!c!} = \frac{a!}{(a-b+c)!(a-b)!}\frac{(a-b+c)!}{(b-c)!c!} = \binom{a}{b-c}\binom{a-b+c}{c}$

Contradiction

1. Suppose for the sake of contradiction that $\sqrt[3]{2}$ is not irrational. Therefore there exist integers a and b for which $\sqrt[3]{2} = \frac{a}{b}$. Cubing both sides, we get $2 = \frac{a^3}{b^3}$. From this, $a^3 = b^3 + b^3$, which contradicts Fermats Last Theorem.

2. Suppose for the sake of contradiction that $a, b \in \mathcal{Z}$ but $a^2 - 4b - 3 \neq 0$. Then we have $a^2 = 4b + 3 = 2(2b+1) + 1$, which means a^2 is odd. Therefore a is odd also, so $a = 2c + 1$ for some integer c. Plugging this back into $a^2 - 4b = 0$ gives us

$$
\begin{aligned}
(2c+1)^2 - 4b - 3 &= 0 \\
4c^2 + 4c + 1 - 4b - 3 &= 0 \\
4c^2 + 4c - 4b &= 2 \\
2c^2 + 2c - 2b &= 1 \\
2(c^2 + c - b) &= 1
\end{aligned}
$$

   From this last equation, we conclude that 1 is an even number, a contradiction.

3. Suppose for the sake of contradiction that a is rational and ab is irrational and b is **not** irrational. Thus we have a and b rational, and ab irrational. Since a and b are rational, we know there are integers c, d, e, f for which $a = \frac{c}{d}$ and $b = \frac{e}{f}$. Then $ab = \frac{ce}{df}$, and since both ce and df are integers, it follows that ab is rational. But this is a contradiction because we started out with ab irrational.

4. Suppose for the sake of contradiction that there do exist integers a and b for which $18a + 6b = 1$. Then $1 = 2(9a + 3b)$, which means 1 is even, a contradiction.

5. Given any collection of 5 consecutive integers, at least one must be a multiple of two, at least one must be a mulitple of three, at least one must be a multiple of four and at least one must be a multiple of five. Hence the product is a multiple of $5 \cdot 4 \cdot 3 \cdot 2 = 120$. In particular, the product is a multiple of 60.

6. **Claim 1**: We say that a point $P = (x, y)$ in $\mathcal{R}$ is **rational** if both x and y are rational. More precisely, P is rational if $P = (x, y) \in \mathcal{Q}^2$. An equation $F(x, y) = 0$ is said to have a **rational point** if there exists $x_0, y_0 \in \mathcal{Q}$ such that $F(x_0, y_0) = 0$. For example, the curve $x^2 + y^2 - 1 = 0$ has rational point $(x_0, y_0) = (1, 0)$. Show that the curve $x^2 + y^2 - 3 = 0$ has no rational points.

   Claim 2: C1 involved showing that there are no rational points on the curve $x^2 + y^2 - 3 = 0$. Use this fact to show that $\sqrt{3}$ is irrational.

   Claim 3: Explain why $x^2 + y^2 - 3 = 0$ not having any rational solutions (C2) implies $x^2 + y^2 - 3^k = 0$ has no rational solutions for k an odd, positive integer.

   Claim 4: Use the above result to prove that $\sqrt{3^k}$ is irrational for all odd, positive k.

   Hints for C1-C4. For C1, first show that the equation $a^2 + b^2 = 3c^2$ has no solutions (other than the trivial solution $(a, b, c) = (0, 0, 0)$) in the integers. To do this, investigate the remainders of a sum of squares (mod 4). After youve done this, prove that the only solution is indeed the trivial solution.

   Now, assume that the equation $x^2 + y^2 - 3 = 0$ has a rational solution. Use the definition of rational numbers to yield a contradiction.

Contrapositive

1. Suppose n is not even. Then n is odd, so $n = 2a + 1$ for some integer a, by definition of an odd number. Thus $n^k = (2a + 1)^k = \sum_{i=0}^{k}(2a)^i = \sum_{i=1}^{k}(2a)^i + 1$. Consequently $n^2 = 2b + 1$, where b is the integer $a\sum_{i=0}^{k-1}(2a)^i$, so n^k is odd. Therefore n^k is not even.

2. Suppose it is not the case that a and b are odd. Then, by DeMorgans Law, at least one of a and b is even. Let us look at these cases separately.

   Case 1. Suppose a is even. Then $a = 2c$ for some integer c. Thus $a^2(b^2 - 2b) = (2c)^2(b^2 - 2b) = 2(2c^2(b^2 - 2b))$, which is even.

   Case 2. Suppose b is even. Then $b = 2c$ for some integer c. Thus $a^2(b^2 - 2b) = a^2((2c)^2 - 2(2c)) = 2(a^2(2c^2 - 2c))$, which is even.

3. Suppose it is not the case that $x < 0$, so $x \geq 0$. Then neither x^2 nor $5x$ is negative, so $x^2 + 5x \geq 0$. Thus it is not true that $x^2 + 5x < 0$.

4. (Direct) Suppose n is odd, so $n = 2a+1$ for some integer a. Then $n^2 - 1 = (2a+1)^2 - 1 = 4a^2 + 4a = 4(a^2 + a) = 4a(a + 1)$. So far we have $n^2 - 1 = 4a(a + 1)$, but we want a factor of 8, not 4. But notice that one of a or $a + 1$ must be even, so $a(a + 1)$ is even and hence $a(a + 1) = 2c$ for some integer c. Now we have $n^2 - 1 = 4a(a + 1) = 4(2c) = 8c$. But $n^2 - 1 = 8c$ means $8|(n^2 - 1)$.

5. Assume n is not prime. Write $n = ab$ for some $a, b > 1$. Then $2^n - 1 = 2^{ab} - 1 = (2^b - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 1)$. Hence $2^n - 1$ is composite.

6. (Direct) Suppose $a \equiv b \pmod{n}$. This means $n|(a-b)$, so there is an integer d for which $a-b = nd$. Multiply both sides of this by c to get $ac - bc = ndc$. Consequently, there is an integer $e = dc$ for which $ac - bc = ne$, so $n|(ac - bc)$ and consequently $ac \equiv bc \pmod{n}$.

Proof by Cases

1. (a) $x > 0, y > 0$. Then $|x||y| = x \cdot y$ and $|xy| = xy$.
   (b) $x > 0, y \leq 0$ Then $|x||y| = x \cdot (-y)$ and $|xy| = -xy$.
   (c) $x \leq 0, y > 0$ Then $|x||y| = -x \cdot y$ and $|xy| = -xy$.
   (d) $x \leq 0, y \leq 0$ Then $|x||y| = -x \cdot (-y) = xy$ and $|xy| = xy$.

2. (a) $a \geq b$. Then $\max(x, y) + \min(x, y) = x + y$.
   (b) $a < b$. Then $\max(x, y) + \min(x, y) = y + x$.

3. First we factor $n^7 - n = n(n^6 - 1) = n(n^3 - 1)(n^3 + 1) = n(n - 1)(n^2 + n + 1)(n + 1)(n^2 - n + 1)$. Now there are 7 cases to consider, depending on $n = 7q + r$ where $r = 0, 1, 2, 3, 4, 5, 6, 7$.

   Case 1: $n = 7q$. Then $n^7 - n$ has the factor n, which is divisible by 7.

   Case 2: $n = 7q + 1$. Then $n^7 - n$ has the factor $n - 1 = 7q$.

   Case 3: $n = 7q + 2$. Then the factor $n^2 + n + 1 = (7q + 2)^2 + (7q + 2) + 1 = 49q^2 + 35q + 7$ is clearly divisible by 7.

Case 4: $n = 7q + 3$. Then the factor $n^2 - n + 1 = (7q+3)^2 - (7q+3) + 1 = 49q^2 + 35q + 7$ is clearly divisible by 7.

Case 5: $n = 7q + 4$. Then the factor $n^2 + n + 1 = (7q+4)^2 + (7q+4) + 1 = 49q^2 + 63q + 21$ is clearly divisible by 7.

Case 6: $n = 7q + 5$. Then the factor $n^2 - n + 1 = (7q+5)^2 - (7q+5) + 1 = 49q^2 + 63q + 21$ is clearly divisible by 7.

Case 7: $n = 7q + 6$. Then the factor$n + 1 = 7q + 7$ is clearly divisible by 7.

4. To compare $|a + b|$ and $|a| + |b|$ is equivalent with comparing $|a + b|^2$ and $(|a| + |b|)^2$.

$$|a + b|^2 = a^2 + 2ab + b^2$$
$$(|a| + |b|)^2 = a^2 + 2|a||b| + b^2$$

Then we check all the four possible case

(a) $a > 0, b > 0$
(b) $a > 0, b \leq 0$
(c) $a \leq 0, b > 0$
(d) $a \leq 0, b \leq 0$

It is easy to see that in all cases, $2ab \leq 2|a||b|$, thus $|a + b| \leq |a| + |b|$

Induction

1. The proof is by induction.

(a) When $n = 1$ the statement is $F_1 = F_{1+2} - 1 = F_3 - 1 = 2 - 1 = 1$, which is true. Also when $n = 2$ the statement is $F_1 + F_2 = F_{2+2} - 1 = F_4 - 1 = 3 - 1 = 2$, which is true as $F_1 + F_2 = 1 + 1 = 2$.

(b) Now assume $k \geq 1$ and $F_1 + F_2 + F_3 + F_4 + \ldots F_k = F_{k+2} - 1$. We need to show $F_1 + F_2 + F_3 + F_4 + \ldots F_k + F_{k+1} = F_{k+3} - 1$. Observe that

$$\begin{aligned} F_1 + F_2 + F_3 + F_4 + \ldots F_k + F_{k+1} &= \\ (F_1 + F_2 + F_3 + F_4 + \ldots F_k) + F_{k+1} &= \\ F_{k+2} - 1 + F_{k+1} &= F_{k+3} - 1 \end{aligned}$$

2. Hint: Use induction on the integer n. After doing the basis step, break up the expression $\binom{k}{r}$ as $\binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}$. Then regroup, use the induction hypothesis, and recombine using the above identity.

3. (Strong Induction) For $n = 1$ this is $\binom{1}{0} + \binom{0}{1} = 1 + 0 = 1 = F_2 = F_{1+1}$. Thus the assertion is true when $n = 1$.

Now fix n and assume that $\binom{k}{0}+\binom{k-1}{1}+\binom{k-2}{2}+\binom{k-3}{3}+\cdots+\binom{1}{k-1}+\binom{0}{k}=F_{k+1}$ whenever $k<n$. In what follows we use the identity $\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$. We also often use $\binom{a}{b}=0$ whenever it is untrue that $0\leq b\leq a$.

$$\begin{array}{rc} & \binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{1}{n-1}+\binom{0}{n} \\ = & \binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{1}{n-1} \\ = & \binom{n-1}{-1}+\binom{n-1}{0}+\binom{n-2}{0}+\binom{n-2}{1}+\binom{n-3}{1}+\binom{n-3}{2}++\cdots+\binom{0}{n-1}+\binom{0}{n} \\ = & \binom{n-1}{0}+\binom{n-2}{0}+\binom{n-2}{1}+\binom{n-3}{1}+\binom{n-3}{2}++\cdots+\binom{0}{n-1}+\binom{0}{n} \\ = & \left[\binom{n-1}{0}+\binom{n-2}{1}+\cdots+\binom{0}{n}\right]+\left[\binom{n-2}{0}+\binom{n-3}{1}+\cdots+\binom{0}{n}\right] \\ = & F_n+F_{n-1} \\ = & F_n \end{array}$$

This completes the proof.

4. When $n=2$, we have $(1-\frac{1}{2})=\frac{1}{2}$ holds.

   Assume it holds for $n=k$. When $n=k+1$,

$$(1-\frac{1}{2})(1-\frac{1}{3})\dots(1-\frac{1}{n-1})(1-\frac{1}{n})=\frac{1}{k}(1-\frac{1}{k+1})=\frac{1}{k}\frac{k}{k+1}=\frac{1}{k+1}=\frac{1}{n}$$

   This complete the proof.

5. When $n=1$, $\frac{1}{1}=1\leq 2-\frac{1}{1}$ holds.

   Assume it holds for $n=k$, when $n=k+1$,

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n-1)^2}+\frac{1}{n^2}\leq 2-\frac{1}{n-1}+\frac{1}{n^2}\leq 2-\frac{1}{n}$$

   The last inequality holds because

$$\begin{array}{rcl} n^2-1 & < & n^2 \\ \frac{n+1}{n^2} & < & \frac{1}{n-1} \\ \frac{1}{n}+\frac{1}{n^2} & < & \frac{1}{n-1} \\ 2-\frac{1}{n-1}+\frac{1}{n^2} & < & 2-\frac{1}{n} \end{array}$$

6. (a) First we find the smallest n such that for all m n, m cents of postage can be produced by using only 3 cent and 7 cent stamps. By trial and errors, we find that we can form m cents of postage for $m=3,6,7,9,10,12,13$ and 14 so we conjecture that the smallest such n is 12.

   Let $P(m)$ be the predicate We can form m cents of postage by using only 3 cent and 7 cent stamps. We try to prove that $P(m)$ is true for all $m\geq 12$ by induction.

   Inductive hypothesis: Assume that $P(k)$ is true for all integer k where $14\leq k\leq m$.

Inductive step: Now we want to produce $m+1$ cent of postage. But $m+1=(m-2)+3$ and by the inductive hypothesis, we can form $m-2$ cents of postage by using only 3cent and 7cent stamps. So we can use one more 3cent stamp in addition to the number of 3cent and 7cent stamps that we use to form $m-2$ cents of postage to form $m+1$ cents of postage.

So by the principle of induction, we can form m cents of postage for any integer $m \geq 12$.

Together with the amounts of postage smaller than 14 that we can form using 3cent and 7cent stamps, the entire set of postage that can be formed is $\{3, 6, 7, 9, 10 \text{ and all } n \geq 12\}$

(b) We need at least two 7cent stamps, otherwise we cannot produce 14 cents of postage. Notice that we can show that two 7cent stamps is sufficient for producing the same set of postage by using almost exaclty the same proof as shown above.

The only major change that we need to make is to modify $P(m)$ to be We can form m cents of postage by using only 3cent stamps and at most two 7cent stamps

7. Make the assumption that $a_k = 2^k$ for all $0 \leq k \leq n$, and prove that for $k+1$.

8. Figure 1 establish one way to constructure the ternary gray code, by reflection, step by step.

Table 2.5 Generation of a larger word-length ternary Gray code.

One-digit ternary code	Two-digit ternary code		Three-digit ternary code	
0	0	00	00	000
1	1	01	01	001
2	2	02	02	002
	2	12	12	012
	1	11	11	011
	0	10	10	010
	0	20	20	020
	1	21	21	021
	2	22	22	022
			22	122
			21	121
			20	120
			10	110
			11	111
			12	112
			02	102
			01	101
			00	100
			00	200
			01	201
			02	202
			12	212
			11	211
			10	210
			20	220
			21	221
			22	222

Figure 1: ternary gray code

Describe it by yourself to form a complete solution recursively.

9. Assume the is solution for the equation

By WOP, we pick the solution $(u, v, w, x) = (a, b, c, d)$ such that (a, b, c, d) has the smallest value of u among all the solutions to the equation.

$$8a^4 + 4b^4 + 2c^4 = d^4$$

Since all the term of left hand side is even, d must be even, so

$$8a^4 + 4b^4 + 2c^4 = (2d')^4$$

where (2d') = d.

Divide both sides by 2 we get

$$4a^4 + 2b^4 + c^4 = 8d'^4$$

Here c must be even, again we can rewrite the equation as

$$4a^4 + 2b^4 + (2c')^4 = 8d'^4$$

where $2c' = c$

Divide both sides by 2 and we get

$$2a^4 + b^4 + 8c'^4 = 4d'^4$$

Apply the same technique to b and a, we will get

$$8a'^4 + 4b'^4 + 2c'^4 = d'^4$$

where $2a' = a, 2b' = b, 2c' = c, 2d' = d$.

Observe that (a, b, c, d) is now another set of solution to the equation and $a < a$.

This contradicts the claim that (a, b, c, d) is the solution set with the smallest value of u.

The assumption is false, hence the equation has no nonzero positive integer solution.

10. It can't be 2.

    Noticing that $1 + 2 + 3 + 4 + 5 = 15$, which is an odd number.

    Every step will not change the parity of the sum, i.e. after 4 times, the only number must be an odd number too.

11. Let $P(k)$ be the hypothesis is true for $n = k$. When $n = 0$, L.S. $= F_0 = 0$ and R.S. $= (\dots)^{-1} \geq 0$. $P(0)$ is true. When $n = 1$, L.S. $= F_1 = 1$ and R.S. $= (\dots)^0 = 1$. $P(1)$ is true. Assume $P(j)$ is true for all j, $0 \leq j \leq k, k \geq 1$.

    $F_{k+1} = F_k + F_{k-1} \leq (\frac{1+\sqrt{5}}{2})^{k-1} + (\frac{1+\sqrt{5}}{2})^{k-2} = (\frac{1+\sqrt{5}}{2})^{k-2}(\frac{1+\sqrt{5}}{2} + 1) = (\frac{1+\sqrt{5}}{2})^k$

    By induction $F_n \leq (\frac{1+\sqrt{5}}{2})^{n-1}$ for all $n \geq 0$.

12. When $n = 1$, L.S. $= x$ and R.S.$= \frac{x-2x^2+x^3}{(1-x)^2} = x$. hence the proposition is true for $n = 1$.

    Assume the proposition is true for $n = k$, i.e.

    $$\sum_{i=1}^{k} ix^i = \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2}$$

    Now consider $n = k + 1$,

    $$\sum_{i=1}^{k+1} ix^i = \sum_{i=1}^{k} ix^i + (k+1)x^{k+1}.$$

    Replace the first term by assumption, we can prove it is true for all $n \geq 1$.

13. It is easy to see that it is possible to tile 6×3 board and 6×2 board using L-shaped pieces. Now assume a $6 \times j$ board can be tiled with L pieces for all j, $3 \leq j \leq k$. Then a $6 \times (k+1)$ board can be split into one $6 \times (k-1)$ board and one 6×2 board. Both of them can be tiled, as a result a $6 \times (k+1)$ board can also be tiled. Hence any $6 \times n$ board can be tiled using L-shaped pieces for any $n \geq 2$.

14. (a) If we are going to prove by induction on value of the fraction f, then what is the base case? It is not possible to come up with a valid base case value. In induction examples we learned, the base case for induction is the smallest possible value of n (usually 0 or 1). But there does not exist such a smallest fraction. This is because given any fraction f', there always exists another fraction f00 such that $f'' < f'$.

    (b) Here we prove by induction on the value of numerator of fraction f. Let the proposition $P(m)$ be "For all $n > 0$, given a fraction $\frac{v}{n}$ for any $1 \leq v \leq m$, the algorithm will finish in finite number of steps."

    The base case is $m = 1$. Obviously, given any fraction $f = \frac{1}{n}$ for any $n > 0$, the new fraction f' must be zero. Hence $P(1)$ is true.

    Now assume $P(k-1)$ is true, consider $n = k$. Given any fraction $f = \frac{k}{n}$, $n > 0$, by the algorithm we get the new fraction f' where

$$f' = \frac{k}{n} - \frac{1}{\lceil n/k \rceil}$$

    Note we may write $\lceil n/k \rceil$ as $\frac{n+j}{k}$ where $0 \leq j < k$. Also $n + j$ is divisible by k, i.e. $n + j = kv$ for some integer v.

$$\begin{aligned} &= \frac{k}{n} - \frac{1}{(n+j)/k} \\ &= \frac{k}{n} - \frac{k}{n+j} \\ &= \frac{k(n+j)-kn}{n(n+j)} \\ &= \frac{kj}{n(n+j)} \\ &= \frac{j}{nv} \end{aligned}$$

    Since $j < k$, the numerator of f' is smaller then k. By assumption, the algorithm with finish in finite number having f' as input. So the same also holds for f because it takes one more step to process f than f'. Hence $P(k)$ is also true. Therefore the proposition is true for all $m > 0$.

15. **Solution**

    (a) Let '$x \to y$' denote'move the top disk of pole x to pole y'. The fastest way to move 2 disks is $1 \to 2, 2 \to 3, 1 \to 2, 3 \to 2, 2 \to 1, 2 \to 3, 1 \to 2, 2 \to 3$.

    (b) In order to move the largest disk from pole 1 to pole 3, we have to first move the disk from pole 1 to pole 2, and then from pole 2 to pole 3.

    In short, the optimal movement for n disks is as follows:

    i. $n - 1$ disks, $1 \to 3$.

ii. largest disk, $1 \to 2$.
iii. $n-1$ disks, $3 \to 1$.
iv. Largest disk, $2 \to 3$.
v. $n-1$ disks, $1 \to 3$.

By symmetry, moving $n-1$ disks from pole 3 to pole 1 uses exactly the same number of steps as moveing the disks from pole 1 to pole 3. Therefore a recurrence relation of R is

$$R(n) = R(n-1) + 1 + R(n-1) + 1 + R(n-1) = 3R(n-1) + 2.$$

The base case is $R(1) = 2$.
Finally,

$$R(n) + 1 = 3(R(n-1) + 1) = 3^2(R(n-2) + 1) = \cdots = 3^{n-1}(R(1) + 1) = 3^n,$$

thus $R(n) = 3^n - 1$.

(c) When $n = 1$, the statement is true as there are only 3 configurations: the disk is at pole 1, 2, and 3, and my strategy passes through all of them.
Suppose the statement is true when $n = k$. For $n = k + 1$, our movement is as follows:

i. k disks, $1 \to 3$. By induction hypothesis, these movements passes through all configurations with largest disk at pole 1.
ii. largest disk, $1 \to 2$.
iii. k disks, $3 \to 1$. By induction hypothesis, these movements passes through all configurations with largest disk at pole 2.
iv. Largest disk, $2 \to 3$.
v. k disks, $1 \to 3$. By induction hypothesis, these movements passes through all confi gurations with largest disk at pole 3.

Therefore the movement passes through all possible configurations when $n = k + 1$.

16. Since $K_0 = 1 \geq 0$, natually, we suppose our assumption to be $K_n \geq n$ holds directly.
However, by assumption we could only know that

$$\begin{aligned} 2K_{\lfloor n/2 \rfloor} &\geq 2\lfloor n/2 \rfloor \\ 3K_{\lfloor n/3 \rfloor} &\geq 3\lfloor n/3 \rfloor \end{aligned}$$

Such equations are not enough. For example, when $n = 2m + 1$,

$$2K_{\lfloor n/2 \rfloor} \geq 2\lfloor n/2 \rfloor = 2m$$

Notice that

$$K_{n+1} = \min\{1 + 2K_{\lfloor n/2 \rfloor}, 1 + 3K_{\lfloor n/3 \rfloor}\}$$

Thus $1+2K_{\lfloor n/2 \rfloor} \geq 1+2m = n$ is not sufficient to demonstrate that $K_{n+1} \geq n+1$. Our induction fails.

We shall prove a “stronger” conclusion than the problem ask:

$$K_n \geq n+1, \quad \text{for } n \geq 0$$

The basis is trival.

By the property of floor function (Check it yourself),

$$m\lfloor n/m \rfloor \geq n - m + 1, \quad n, m \in \mathcal{Z}^+$$

Now that

$$2K_{\lfloor n/2 \rfloor} \geq 2(\lfloor n/2 \rfloor + 1) = 2\lfloor n/2 \rfloor + 2 \geq (n-1) + 2 = n+1$$

and

$$3K_{\lfloor n/3 \rfloor} \geq 3(\lfloor n/3 \rfloor + 1) = 3\lfloor n/3 \rfloor + 3 \geq (n-2) + 3 = n+1$$

Thus

$$K_{n+1} = 1 + \min\{2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}\} \geq 1 + (n+1) = n+2$$

At last we have $K_{n+1} \geq (n+1) \geq n$ holds for all nonnegative n.