CSC2110 Discrete Mathematics
Homework 1 solutions

Q1 (a) i. If the audience do not sleep, then the chairperson does not give the lecture.

ii. If you do not have the proper security clearance, then you cannot inspect the aircraft.

(b) $$\begin{aligned}\neg(\exists x(Q(x) \rightarrow \forall y P(y))) &\equiv \forall x \neg(Q(x) \rightarrow \forall y P(y)) \\ &\equiv \forall x \neg(\neg Q(x) \vee \forall y P(y)) \\ &\equiv \forall x(Q(x) \wedge \exists y \neg P(y))\end{aligned}$$

(c) Name the given facts as follows:

i. $\neg q \rightarrow P(a)$

ii. $q \rightarrow Q(b)$

iii. $s \vee \neg \exists x Q(x)$

iv. $\forall x \neg P(x)$

Steps of deduction:

1. By (iv), for any a, $\neg P(a)$ is true.
2. By (i), apply contrapositive and we get q is true.
3. By (ii), $Q(b)$ is true.
4. By (iii), since $\neg \exists x Q(x)$ is false, s must be true.

(d) Let A be "John is lying", J be "Jesse is lying", T be "Tom is lying" and etc. Then by the given facts, we have

i. $T \wedge J \rightarrow \neg L$

ii. $T \vee H \rightarrow J$

iii. $\neg T \rightarrow A$

iv. $\neg A$

v. $L \vee H$

Steps of deduction:

1. By (iv), A is false.
2. By (iii), applying contrapositive we get T is true.
3. By (ii), we get J is also true.
4. By (i), L is false.
5. By (v), since L is false H must be true.

To conclude, Tom, Jesse and Hackson are lying.

Q2 (a) $n^4 + 2n^3 - n^2 - 2n = n(n^3 + 2n^2 - n - 2)$
$= n(n-1)(n^2 + 3n + 2)$
$= (n-1)n(n+1)(n+2)$

For any 4 consecutive integers, one of them must be divisible by 4. Hence the above expression is divisible by 4.

(b) Assume $\sqrt{10}$ is rational, then we can write $\sqrt{10} = \frac{m}{n}$ such that m and n are both integers and they do not have common factors other than 1. Then we get

$$10 = \frac{m^2}{n^2}$$
$$10n^2 = m^2$$

Since left hand side is even, m must be even, substitue $m = 2k$

$$10n^2 = (2k)^2$$
$$5n^2 = 2k^2$$

Since right hand side is even, n must be even. Hence m and n are both even, which contradicts the assumption that m and n do not have common factor other 1. Hence $\sqrt{10}$ is irrational.

(c) Let $x = a^2 + b^2$ and $y = c^2 + d^2$. Then

$xy = (a^2 + b^2)(c^2 + d^2)$
$= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$
$= (a^2c^2 + b^2d^2 + 2abcd) + (a^2d^2 + b^2c^2 - 2abcd)$
$= (ac + bd)^2 + (ad - bc)^2$

(d) Assume there exists another super prime tuple $(x, x+2, x+4)$ other than $(3, 5, 7)$. Consider $k = x \mod 3$, then if

i. $k = 0$, then x is not a prime. Contradiction.

ii. $k = 1$, then $(x + 2 \mod 3) = (1 + 2 \mod 3) = 0$, which implies $x + 2$ is not a prime. Contradiction.

iii. $k = 2$, then $(x + 4 \mod 3) = (2 + 4 \mod 3) = 0$, which implies $x + 4$ is not a prime. Contradiction.

Hence the assumption is false, and $(3, 5, 7)$ is the only super prime tuple.

Q3 (a) Let $P(k)$ be the hypothesis is true for $n = k$.
When $n = 0$, L.S. $= F_0 = 0$ and R.S. $= (...)^{-1} \geq 0$. $P(0)$ is true.
When $n = 1$, L.S. $= F_1 = 1$ and R.S. $= (...)^0 = 1$. $P(1)$ is true.
Assume $P(j)$ is true for all j, $0 \leq j \leq k$, $k \geq 1$.

$$
\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&\leq (\frac{1+\sqrt{5}}{2})^{k-1} + (\frac{1+\sqrt{5}}{2})^{k-2} \\
&= (\frac{1+\sqrt{5}}{2})^{k-2}(\frac{1+\sqrt{5}}{2} + 1) \\
&= (\frac{1+\sqrt{5}}{2})^{k-2}(\frac{3+\sqrt{5}}{2}) \\
&= (\frac{1+\sqrt{5}}{2})^{k-2}(\frac{1+\sqrt{5}}{2})^2 \\
&= (\frac{1+\sqrt{5}}{2})^{k}
\end{aligned}
$$

$\therefore$ By induction $F_n \leq ((1+\sqrt{5})/2)^{n-1}$ for all $n \geq 0$.

(b) When $n = 1$, L.S. $= x$ and R.S. $= \frac{x-2x^2+x^3}{(1-x)^2} = x$, hence the proposition is true for $n = 1$.
Assume the proposition is true for $n = k$, i.e.

$$\sum_{i=1}^{k} ix^i = \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2}$$

Now consider $n = k+1$, $\quad \sum_{i=1}^{k+1} ix^i$

$$
\begin{aligned}
&= \sum_{i=1}^{k} ix^i + (k+1)x^{k+1} \\
&= \frac{x - (k+1)x^{k+1} + kx^{k+2}}{(1-x)^2} + (k+1)x^{k+1} \\
&= \frac{x - (k+1)x^{k+1} + kx^{k+2} + (k+1)x^{k+1}(1-x)^2}{(1-x)^2} \\
&= \frac{x - (k+1)x^{k+1} + kx^{k+2} + (k+1)(x^{k+3} - 2x^{k+2} + x^{k+1})}{(1-x)^2} \\
&= \frac{x - (k+2)x^{k+2} + (k+1)x^{k+3}}{(1-x)^2}
\end{aligned}
$$

$\therefore$ By induction the proposition is true for all $n \geq 1$

(c) It is easy to see that it is possible to tile 6×3 board and 6×2 board using L-shaped pieces. Now assume a $6 \times j$ board can be tiled with L pieces for all j, $3 \leq j \leq k$. Then a $6 \times (k+1)$ board can be split into one $6 \times (k-1)$ board and one 6×2 board. Both of them can be tiled, as a result a $6 \times (k+1)$ board can also be tiled. Hence any $6 \times n$ board can be tiled using L-shaped pieces for any $n \geq 2$.

Q4 (a) If we are going to prove by induction on value of the fraction f, then what is the base case? It is not possible to come up with a valid base case value. In induction examples we learned, the base case for induction is the smallest possible value of n (usually 0 or 1). But there does not exist such a smallest fraction. This is because given any fraction f', there always exists another fraction f'' such that $f'' < f'$.

(b) Here we prove by induction on the value of numerator of fraction f. Let the proposition $P(m)$ be "For all $n > 0$, given a fraction $\frac{v}{n}$ for any $1 \leq v \leq m$, the algorithm will finish in finite number of steps."
The base case is $m = 1$. Obviously, given any fraction $f = \frac{1}{n}$ for any $n > 0$, the new fraction f' must be zero. Hence $P(1)$ is true.
Now assume $P(k-1)$ is true, consider $n = k$. Given any fraction $f = \frac{k}{n}$, $n > 0$, by the algorithm we get the new fraction f' where

$$f' = \frac{k}{n} - \frac{1}{\lceil n/k \rceil}$$

Note we may write $\lceil n/k \rceil$ as $\frac{n+j}{k}$ where $0 \leq j < k$. Also $n + j$ is divisible by k, i.e. $n + j = kv$ for some integer v.

$$\begin{aligned}
&= \frac{k}{n} - \frac{1}{(n+j)/k} \\
&= \frac{k}{n} - \frac{k}{n+j} \\
&= \frac{k(n+j) - kn}{n(n+j)} \\
&= \frac{kj}{n(n+j)} \\
&= \frac{j}{nv}
\end{aligned}$$

Since $j < k$, the numerator of f' is smaller then k. By assumption, the algorithm with finish in finite number having f' as input. So the same also holds for f because it takes one more step to process f than f'. Hence $P(k)$ is also true. Therefore the proposition is true for all $m > 0$.