CSC2110 Discrete Mathematics **Last updated: 6-Oct**
Homework 1 **Due Date: 19-Oct-2009 (in class)**

Total mark of this homework is 25. You are encouraged to collaborate on the homework, but you must write your own solutions and list your collaborators on your solution sheet (you will not lose any mark by doing this).

Q1 (6 marks)

(a) **(1 mark)** Write contrapositive for each of the following statements

  i. The audience will go to sleep if the chairperson gives the lecture.

  ii. You may inspect the aircraft only if you have the proper security clearance.

(b) **(1 mark)** Apply DeMorgan's law to $\neg(\exists x(Q(x) \rightarrow \forall y P(y)))$

(c) **(2 marks)** What conclusion can you make, given the followings are true?

- $\neg q \rightarrow P(a)$
- $q \rightarrow Q(b)$
- $s \vee \neg \exists x Q(x)$
- $\forall x \neg P(x)$

(d) **(2 marks)** A detective has interviewed five witnesses to a crime. From their stories, the detective has concluded that:

- If both Tom and Jesse are lying then Leo is telling the truth.
- If Tom or Hackson is lying, then Jesse is also lying.
- If Tom is telling the truth then John is lying.
- John is a well respected teacher so he never lies.
- Either Leo or Hackson is lying.

What conclusion can you make? There may be more than one liar. Show your steps.

Q2 (8 marks)

(a) **(2 marks)** For any integer n, $n^4 + 2n^3 - n^2 - 2n$ is divisible by 4.

(b) **(2 marks)** Prove that $\sqrt{10}$ is irrational.

(c) **(2 marks)** A number n is a sum of two squares if $n = a^2 + b^2$ for some integers a and b. If x and y are both sum of two squares, prove that xy is also a sum of two squares.

(d) **(2 marks)** If the tuple (a, b, c) satisfy the following conditions:

- a, b and c are consecutive odd integers.
- a, b and c are all primes.

Then we call it a super prime tuple. Prove that $(3, 5, 7)$ is the only super prime tuple.

Q3 (6 marks)

(a) **(2 marks)** Define $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n = 0, 1, 2, \ldots$. Prove that for any $n \geq 0$ we have $F_n \leq ((1 + \sqrt{5})/2)^{n-1}$.

(b) **(2 marks)** Prove by induction the following is true.

$$\sum_{i=1}^{n} ix^i = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

(c) **(2 marks)** Show that a $6 \times n$ board ($n \geq 2$) can be tiled with L-shaped tiles, without gap and overlapping. Each L-shaped tiles cover three squares.

Q4 (5 marks)

In ancient Egypt, fractions were written as sums of fractions with numerator 1. For instance, $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$. Consider the following algorithm for writing a fraction $\frac{m}{n}$ in this form ($1 \leq m < n$): write the fraction $\frac{1}{\lceil n/m \rceil}$, calculate the fraction $\frac{m}{n} - \frac{1}{\lceil n/m \rceil}$, and if it is nonzero repeat the same step. Here $\lceil n/m \rceil$ denotes the smallest integer not less than n/m.

(a) **(2 marks)** Hackson tries to prove that the algorithm always finishes in a finite number of steps. Below is a sketch of his proof:

> Observe that given a fraction f to the algorithm, value of f will be decreasing as the algorithm goes on. In other words, the new fraction f', obtained from f after applying one step of the algorithm, will be smaller then f. If I prove by induction on value of f, then I can assume applying the algorithm on f' will finish in finite number of steps. Hence the algorithm will finish in finite number of steps having f as input, because having f as input just require one more steps than having f' as input.

What's wrong with the proof?

(b) **(3 marks)** Prove the algorithm always finishes in a finite number of steps using induction.