

Tutorial 11: Elementary Graph Theory

Yuanming Yu
yyming [at] gmail [dot] com

December 5, 2011

1. Show that the Gale-Shapley algorithm is men-optimal, and also women-pessimal.

Proof. Claim: If a woman w rejects a man m during the execution of the Gale-Shapley algorithm, then (m, w) cannot be valid partners.

We prove the claim by induction. In the first iteration, if w rejects a man m , then there is a man m^* who also proposes to w and m^* is higher on w 's list than m . Suppose, by way of contradiction, that (m, w) are matched in a stable matching M . Since w is the first on m^* list (as m^* proposed to w in the first iteration), m^* prefers w to his partner in M . This implies that (m^*, w) is an unstable pair in M , which contradicts that M is a stable matching.

Now assume the claim is true for the k -th iteration, we prove that it is also true for the $k + 1$ -th iteration. The argument is almost the same as above. If w rejects a man m in the $k + 1$ -th iteration, then there is a man m^* who also proposes to w in the $k + 1$ -th iteration and m^* is higher on w 's list than m . Suppose (m, w) are matched in a table matching M . By induction hypothesis, the women that have rejected m^* (in or before the k -th iteration) cannot be valid partners for m^* . Since m^* proposed in a non-increasing order, m^* prefers w to his partner in M . This implies that (m^*, w) is an unstable pair in M , which contradicts that M is a stable matching.

Now goes our main proof: Gale-Shapley algorithm is men-optimal, and also women-pessimal.

Since a man proposes in a non-increasing order, by Claim, the first woman who does not reject him is his best valid partner. Similarly, since the sequence of proposals that a woman is holding is non-decreasing, by Claim, the first man she does not reject is her worst valid partner. \square

2. Show an example that women can lie to get a better partner.

Someone asked how cheating is possible in the stable marriage problem. Here is an example of 3 men (a,b,c) and 3 women (1,2,3). Suppose the algorithm is the Gale-Shapley algorithm and is men-optimal. Consider the following "true" preference lists.

man a prefers $2 > 1 > 3$

man b prefers $1 > 2 > 3$

man c prefers $2 > 3 > 1$

woman 1 prefers $a > b > c$

woman 2 prefers $b > a > c$

woman 3 prefers $c > b > a$

Then the Gale-Shapley (men-optimal) algorithm would find the matching $(a,2),(b,1),(c,3)$.

Note that woman 2 only gets her second choice if she uses her true preference list.

In fact, woman can get her first choice (man b) if she uses the following "faked" list.

woman 2 prefers $b > c > a$

Now, the Gale-Shapley (men-optimal) algorithm would find the matching $(a,1),(b,2),(c,3)$.

Try it.

3. (a) Think about Stable matching when there are more applicants than positions scenario.
- (b) Generalize the definition of unstable pairs.
- (c) Use the Gale-Shapley result to show that there is a stable matching in this more general setting.

Solution:

- (a) Adding dummy positions with number equal to number of applicant minus number of position, and put those positions at the end of every applicant's preference list;
- (b) the definition of unstable pair is the same as the old one, besides that for every dummy positions, all applicants are the same.
- (c) For dummy position, just set an arbitrary preference list order to every applicants.

Now that the problem can be regarded as a Stable Marriage Problem, by Gale-Shapley there is a stable matching.

Since for every dummy position, there is no difference among applicants, so the specific preference version has no unstable pair imply that the general setting has no unstable pair, thus it is also a stable matching.

4. Show that a bipartite graph has a matching of size $n - k$ if and only if there is no subset S of one side with $|N(S)| + k < |S|$, where n is the number of nodes on one side and so the graph has $2n$ nodes.

Remark: This is a generalization of Hall's theorem.

Solution:

- Reduce the problem to perfect matching by adding dummy nodes x_1, x_2, \dots, x_k on the left side and y_1, y_2, \dots, y_k on the right side.
- dummy nodes connect to every node in the opposite side, i.e. for $u \in \text{left}[G]$, adding (u, y_i) , and for $v \in \text{right}[G]$, adding (x_i, v) , and adding (x_i, y_j) .
- Then there is a perfect matching in the new graph if and only if there is a matching of size $n - k$ in the old graph: 1) the old graph has a matching of size $n - k$, clearly the new one has a matching of size $n + k$ (for the $2k$ dummy nodes can make pair with any node); 2) the new one has of size $n + k$, then delete those pairs has dummy vertices will affect at most $2k$ pairs, thus we have a matching of size $n - k$ in the old graph.

The problem we want to prove is that: a bipartite graph has a matching of size $n - k$ if and only if there is no subset S of one side with $|N(S)| + k < |S|$, which is equivalent to: a bipartite graph has no matching of size $n - k$ if and only if there is a subset S of one side with $|N(S)| + k < |S|$ (contrapositive and the property of iff).

One direction is easier: if there is a set S with $|N(S)| + k < |S|$ then there is no matching of size $n - k$. In fact, if there is a set S with $|N(S)| + k < |S|$, then the whole graph's matching size is at most $|N(S)| + |V - S|$ because at most $|N(S)|$ vertices of S can be matched, and this is at most $|S| - k - 1 + |V - S| = n - k - 1$.

The other direction is harder. if the graph has no matching of size $n - k$, then there is a subset S of one side with $|N(S)| + k < |S|$.

We use the fact that there is a matching of size $n - k$ in the old graph if and only if there is a perfect matching in the new graph. By Hall's theorem, there is no perfect matching in the new graph if and only if there is a set S' with $|N(S')| < |S'|$ in the new graph.

Since the old graph has no matching of size $n - k$, we know that there is set S' with $|N(S')| < |S'|$ in the new graph.

Note that the dummy nodes cannot be in S' , because they connect to everyone in the opposite side, which would make $|N(S')| = n + k$ and further the inequality do not hold.

Also note that the dummy nodes on the other side must be in $N(S')$, because they connect to everyone to the opposite side, and S' is not empty thus there is one node in it connects to all these k dummy nodes.

Delete the dummy nodes on both sides, we should have $S = S'$ (since we proved above that there is no dummy nodes in S') and

$$N(S) = N(S') \setminus \{\text{all } k \text{ dummy nodes on that side}\}$$

so that $|N(S)| + k = |N(S')|$. Then we see that $|N(S)| + k < |S|$, and we are done.

In sum, a bipartite graph has a matching of size $n - k$ if and only if there is no subset S of one side with $|N(S)| + k < |S|$.

5. An example that degree upper bounds could be any positive number.

- n jobs (j_i) each taking one unit of time.
- m machines, machine m_i can run c_i jobs.
- And define happily $\sum_{i=1}^m c_i = n$

- (a) Is it possible to assign jobs to machines so that you only need to wait one unit of time and finish them all?
- (b) Model this as a bipartite matching problem. Show the correspondence (how you can construct an instance of bipartite matching problem given the specs of jobs and machines).

Solution:

- a job $j_i \rightarrow j_i$ on the left
- a machine m_i with capacity $c_i \rightarrow c_i$ vertices $\{m_{i,j} | j = 1, 2, \dots, c_i\}$ on the right.
- each vertex on the right is connected to all vertices on the left
- An assignment equivalent to a (perfect) matching

Picky Machines: Machines & Jobs Again...

- ❶ Not every job can be fed into every machines.
- ❷ E.g. 6 jobs, 3 machines:

machine	capacity	capability
m_1	3	j_1, j_2
m_2	1	j_3, j_4
m_3	2	j_4, j_5, j_6

Is there an assignment that does all your jobs for one unit of time?

Solution

The same model as the previous one. Now verify with Hall's theorem.

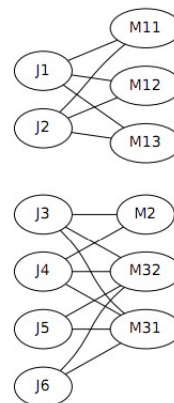


Figure 1: Verify