Tutorial 8: Counting and Functions

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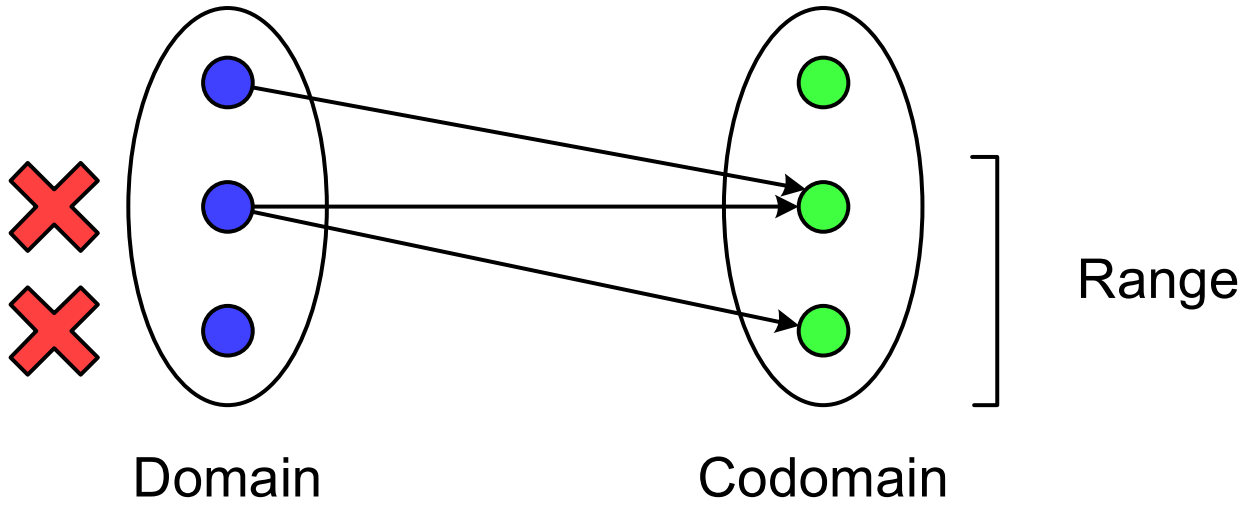
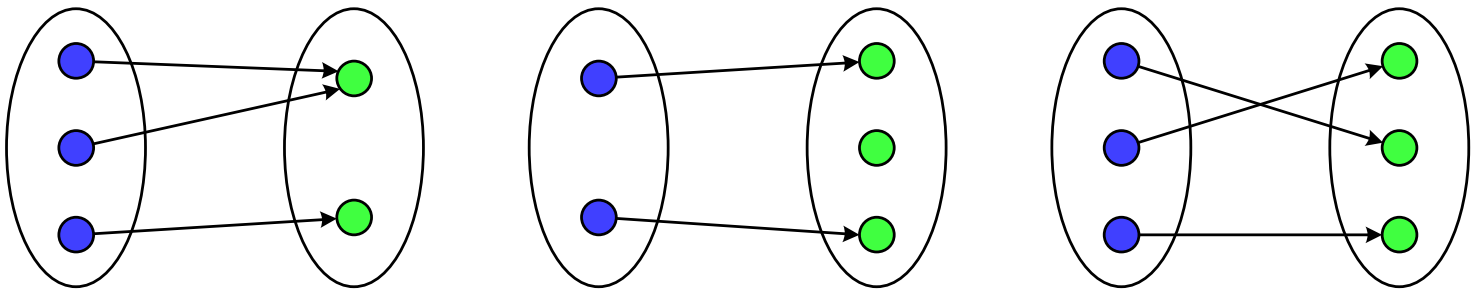
- Functions: every element in the domain has to be mapped (exactly 1 arrow)



- Codomain and range: range contains elements that are mapped to by some element in the domain;
  Codomain can be larger.
- "A function f from set A to set B" is written as $f : A \to B$
- E.g. $f : \mathbb{Z} \to \mathbb{Z}^+$ given by $f(x) = x^2$. Domain $= \mathbb{Z}$, codomain $= \mathbb{Z}^+$, range is the set of "square numbers".

- Surjection (onto): every output is possible;
  every element on the right has ≥ 1 arrow in;
  in this case codomain = range.



- Injection (one-to-one): if $f(a) = f(b)$ then $a = b$;
  every element on the right has ≤ 1 arrow in;
  size of codomain $\geq$ size of domain (if countable).
- Bijection = surjection + injection;
  the function has inverse.
  size of domain, codomain and range are equal (if countable).

T/F Question 1

of functions $f : A \to B$, where $|A| = a$ and $|B| = b$ is a^b.

Solution

- For f to be a function, every element in A has to be mapped to some element in B
- # choices for a particular element in A: b
- # functions: b^a. False
- E.g. # 2-input Boolean functions: 2 (output can be 0 or 1) to the power 4 (4 combinations for 2 inputs) = 16

$\Rightarrow$ In-class exercises on slide 12 of lecture 15. (Composite fn)

T/F Question 1

Given 21 distinct 5-digit positive integers, there are two disjoint subsets which sum to the same number.

Solution (Pigeonhole principle)

- Number of non-empty subsets that can be formed (pigeons / inputs): $2^{21} - 1 = 2097151$.
- Total sum (pigeonholes / outputs) is at most $21 \times 100000 = 2100000$. (Actually a bit less than this)
- So seems the principle cannot be used. . . is it?
- The sum is at least 10000 (5-digit number), so the range is at most 10000–2100000, i.e. 2090001, which is less than 2097151. True
- What if the word "distinct" and "disjoint" are removed? Or one of them?

T/F Question 2

The mean test score for a class of 10 is 6.6. If the test score for each student is an even number from 0 to 10, then there is a student who scores 8 or above.

Solution (Generalized pigeonhole principle)

- Total score of all students: 66 (33 pigeons)
- Distribute the score among the 10 students (pigeonholes)
- By pigeonhole principle, exist a student with

  at least $\left\lceil \frac{33}{10} \right\rceil = 4$ pigeons (8 mark) True
- Averaging argument:

  there is a student with score $\geq$ average.

Why is the formula correct?

If an element x is outside of the union, we want it to be counted 0 times; if it is inside, we want it to be counted exactly 1 time.

- If x belongs to only one set, then it is counted once.
- If x belongs to two sets A_i and A_j, then it is counted $(1\ (A_i) + 1\ (A_j)) - 1\ (A_i \cap A_j) = 1$ time.
- If x belongs to exactly k sets, say $A_1, \ldots, A_5$, (i.e. $k = 5$)
  - Single set $(+)$: $A_1, A_2, \ldots, A_5$ — $5 = \binom{k}{1}$
  - 2-set intersection $(-)$: $A_1 \cap A_2, A_3 \cap A_4, A_1 \cap A_4 \ldots$ (pick any 2 out of the 5) — $\binom{k}{2}$
  - $\ldots$
  - 5-set intersection $(+)$: $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$ — $1 = \binom{k}{k}$
  - 6-set intersection: x is not inside any such intersection — 0
  - $\ldots$ (0 for the rest of the intersections)

Short Question 2

When we sort permutations of the letters ABCDE alphabetically, e.g. ABCDE, ABCED, ABDCE, ABDEC, ABECD, . . . , how many strings come before "BEDAC"? Show steps.

Solution

(Divide the strings before BEDAC into mutually exclusive cases.)

- "A????" comes before "BEDAC": $\# = 4!$
- BA???, BC???, BD???: $\# = 3 \times 3!$
- BEA??, BEC??: $\# = 2 \times 2!$
- BEDA?: the string BEDAC itself
- Total $= 4! + 3 \times 3! + 2 \times 2! = 46$, i.e. the string BEDAC is the 47th string.

Short Question 3

Which is the 21st string when permutations of the string ABCDE is sorted alphabetically? Show steps.

Solution

- Similar to the question before, but in reversed manner
- $21 - 1 = 20 < 4! = 24$: The string should start with "A"
- $20 = 3 \times 3! + 2$: AB???, AC???, AD??? come before it. String starts with "AE"
- $2 = 1 \times 2! + 0!$: "AEC"
- $0 = 0 \times 1!$: The string is "AECBD"

Counting

- Q3, 4, 11, 17, 23, 28 of Set 6.3
- Q3, 6, 24 of Set 6.4

Function

- Q31, 36, 38 of Set 7.1
- Q16, 17 of Set 7.2

Solutions are available in Appendix B of the book.