Tutorial 7: Set Theory and Counting

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T/F Question 1

$\{1, 2, 3\} = \{3, 2, 3, 1\}$.

Solution

- True; order and number of occurrence does not matter

T/F Question 2

The set $A = \{1, \{2, 3\}, 4\}$ contains 4 elements.

Solution

- False; $\{2, 3\}$ is an element of A
- Elements of a set can also be a set

T/F Question 3

$\varnothing \subseteq \text{pow}(\{a\})$.

Solution

- $\text{pow}(\{a\})$ is a set.
- Empty set is a subset of any set.
- So $\varnothing \subseteq \text{pow}(\{a\})$. True
- $\varnothing \in \text{pow}(\{a\})$ is also true, since $\text{pow}(\{a\}) = \{\, \{a\}, \varnothing \,\}$
- But in general, $\varnothing$ is not a member of any set: $\varnothing \notin \{a\}$.

MC Question 1

Let E and O be the set of even numbers and odd numbers respectively. Which of the following is not the set of non-negative odd numbers?

(a) $\{ x \in \mathbb{Z} \mid \exists y \in \mathbb{N}, x = 2y - 1 \}$

(b) $\mathbb{N} \cap \{ x \in \mathbb{Z} \mid x \equiv 1 \bmod 2 \}$

(c) $\mathbb{Z} - E$

(d) $\mathbb{N} \cap O$

Solution

Ans: (c)

(c) is the set of odd numbers. -3 is in $\mathbb{Z} - E$ but it is not a non-negative odd number.

Short Question 1

Prove $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

using set identities

Solution

$$\begin{aligned}
\text{RHS} &= (A \cup B) - (A \cap B) \\
&= ((A \cap B) \cup (A \cap B^c) \cup B) - (A \cap B) \\
&= ((A \cap B) \cup (A \cap B^c) \cup (B \cap A) \cup (B \cap A^c)) - (A \cap B) \\
&= ((A \cap B) \cup (A \cap B^c) \cup (B \cap A^c)) - (A \cap B) \\
&= (A \cap B^c) \cup (B \cap A^c) \qquad (*) \\
&= (A - B) \cup (B - A) \\
&= \text{LHS}
\end{aligned}$$

(*) works since $(A \cap B)$, $(A \cap B^c)$ and $(B \cap A^c)$ are disjoint

Combination vs. permutation

- Combination: order does not matter;
  Permutation: order does matter.
- Permutation: how many ways are there to form a queue of k persons out of n? (an easier question)
- 1st position: n possibilities.
  2nd position: $n - 1$ possibilities.
  ...
  kth position: $n - k + 1$ possibilities.
- (Generalized product rule)
  # permutations $= n(n-1)\cdots(n-k+1)$;
  can also be written as $n!/(n-k)!$.

Combination vs. permutation (cont)

- Permutation: $n!/(n-k)!$.
- Combination: order does not matter.
- $(1,2,3,4)$, $(4,3,2,1)$, $(1,3,2,4)$ are different permutations, but corresponds to the same combination.
- 1 combination can be converted to $k!$ permutations by shuffling the elements inside.
- Hence # combinations = # permutations / $k!$:
  $\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$.

MC Question 1

Consider the following training program: At week 1, I lift a weight of 5 kg. Every week after, I increase the weight by 1 kg or 2 kg. My goal is to be able to lift a weight of 15 kg by the end of week 7. One of the possible programs is: 5, 6, 7, 9, 11, 13, 15. How many such programs are there?

(a) 9 (b) 12 (c) 15 (d) 18

Solution

- The program must consist of a sequence of 1's and 2's, with exactly 2 1's and 4 2's.
- Answer is $\binom{6}{2} = 15$. (c)

Short Question 1

Evaluate the coefficient of the term x^3y^7 in the expansion of $(x-3y)^{10}$.

Solution (Binomial Theorem)

- By the binomial formula, $(A+B)^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i}$
- We have $A = x$, $B = -3y$, $n = 10$

  Want the term $i = 3$
- Coefficient $= \binom{10}{3} (1)^3(-3)^7 = -262440$
- Remember the $+/-$ sign

Short Question 2

How many 5-card hands is three of a kind, i.e. 3 cards are of the same rank, and the remaining card does not form a pair?

Solution

- Rank of the three cards: 13 ways
- Suits of the triple: $\binom{4}{3}$ ways
- Select two ranks from the remaining cards: $\binom{12}{2}$ ways
- Suits of the two cards: 4^2 ways
- $\#$ hands with three of a kind $= 13 \times \binom{4}{3} \times \binom{12}{2} \times 4^2 = 54912$

Short Question 3 (Combinatorial Proof)

Proof the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$ using combinatorial proof, where $k \le r \le n$.

Solution

- LHS can be read as # ways to select r persons for award, and then pick k out of the r persons for gold medal (and the rest for silver medal).
- RHS can be read as # ways to first select k persons for gold medal, and out of the rest $n-k$ persons select $r-k$ for silver medal.
- In both cases there are k gold medalist and $r-k$ silver medalist, so $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

Short Question 4

Count the number of integers from 1 to 1000 that is relatively prime to 4 and 10.

Solution (Inclusion-Exclusion principle — 2 sets)

- Since 4 and 10 are not prime, we need to factorize them: $4 = 2^2$, $10 = 2 \times 5$.
- The question is equivalent to: number of integers from 1 to 1000 that is relatively prime to 2 and 5.
- Count the reverse: those who do have 2 or 5 as a factor.
- $|A|$ (# integers, multiple of 2): 500.
- $|B|$ (# integers, multiple of 5): $1000/5 = 200$.
- Need to subtract those in $A \cap B$, i.e. multiple of 10: 100.
- Answer: $1000 - (500 + 200 - 100) = 400$.

Short Question 5

Calculate $\varphi(1001)$, the number of integers from 1 to 1001 that is relatively prime to 1001.

Solution (Inclusion-Exclusion principle — 3 sets)

- First factorize it: 1001 is a multiple of 11 (homework 2)
- $1001 = 7 \times 11 \times 13$. Let A, B and C be the set of integers in the range $[1, 1001]$ that is multiple of 7, 11 and 13 resp.
- $|A| = 11 \times 13 = 143$, $|B| = 7 \times 13 = 91$, $|C| = 7 \times 11 = 77$.
- $|A \cap B| = 13$, $|B \cap C| = 7$, $|C \cap A| = 11$.
- $|A \cap B \cap C| = 1$.
- So $\varphi(1001) = 1001 - (143 + 91 + 77 - 13 - 7 - 11 + 1) = 720$.