CMSC 5743

Efficient Computing of Deep Neural Networks

Lecture 04: Low Rank Decomposition

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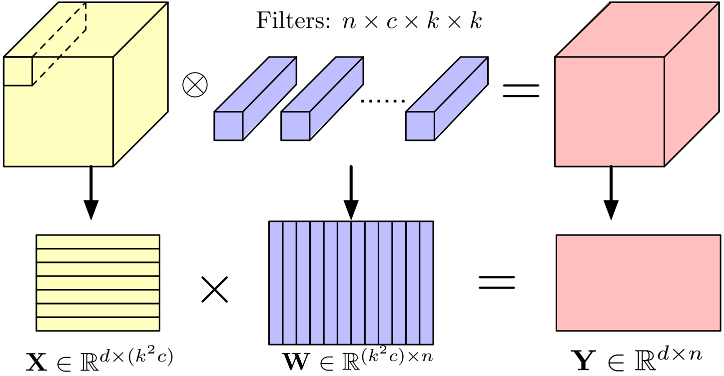
Overview

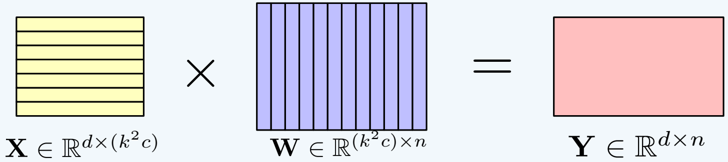
Overview

`Im2col` (Image2Column) Convolution



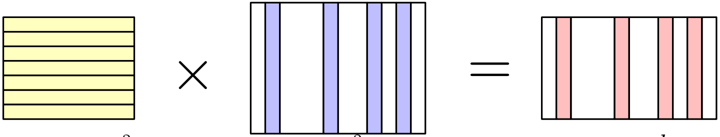
- Transform convolution to matrix multiplication
- Unified calculation for both convolution and fully-connected layers

Matrix Approximation or Matrix Regression?



- Matrix approximation: $\boldsymbol{W} \approx \boldsymbol{W'}$
- Matrix regression: $\boldsymbol{Y} = \boldsymbol{W} \cdot \boldsymbol{X} \approx \boldsymbol{W'} \cdot \boldsymbol{X}$

Compression Approach 1: Sparsity



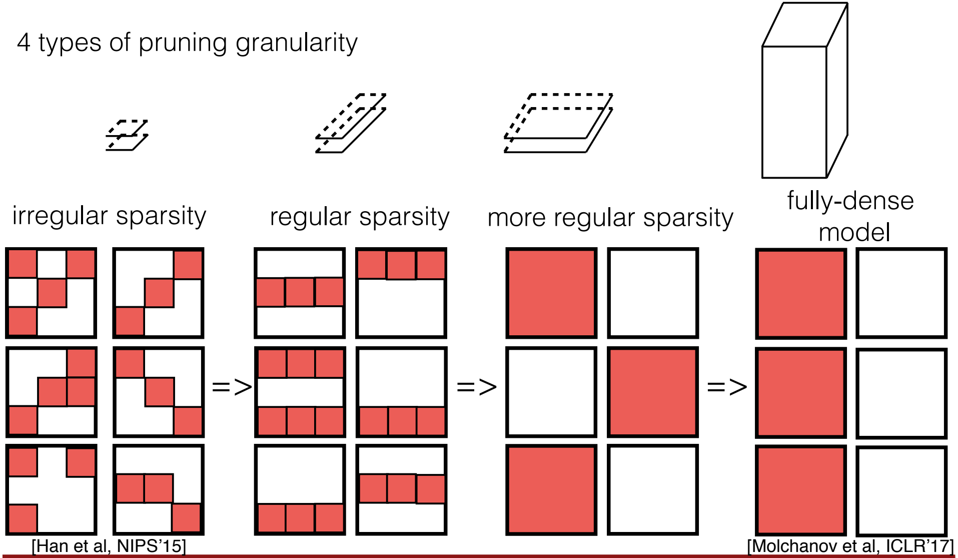
$\mathbf{X} \in \mathbb{R}^{d \times (k^2 c)}$ $\times$ $\mathbf{S} \in \mathbb{R}^{(k^2 c) \times n}$ $=$ $\mathbf{Y} \in \mathbb{R}^{d \times n}$

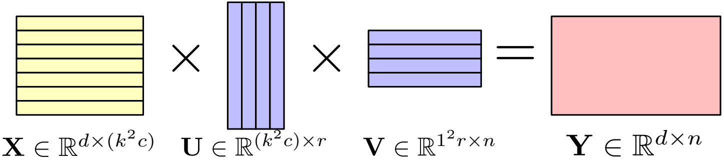
Sparse DNN

- *Sparsification*: weight pruning;
- *Compression*: compressed sparse format for storage;
- *Potential acceleration*: sparse matrix multiplication algorithm.

Exploring the Granularity of Sparsity that is Hardware-friendly

4 types of pruning granularity

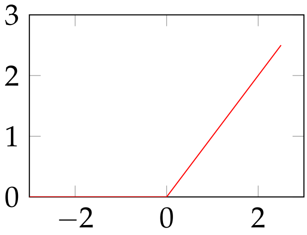




Low-rank DNN

- *Low-rank approximation*: matrix decomposition or tensor decomposition.
- *Compression and acceleration*: less storage required and less FLOP in computation.

Non-linearity Approximation



ReLU

- Activation unit: `ReLU`
- Error more sensitive to positive response;
- Enlarge the solution space.

$$\min_{\boldsymbol{W}} \sum_{i=1}^{N} \|\boldsymbol{W}\boldsymbol{X}_i - \boldsymbol{Y}_i\|_F \rightarrow \min_{\boldsymbol{W}} \sum_{i=1}^{N} \|r(\boldsymbol{W}\boldsymbol{X}_i) - \boldsymbol{Y}_i\|_F$$

- $\boldsymbol{X}$: input feature map
- $\boldsymbol{Y}$: output feature map

Overview

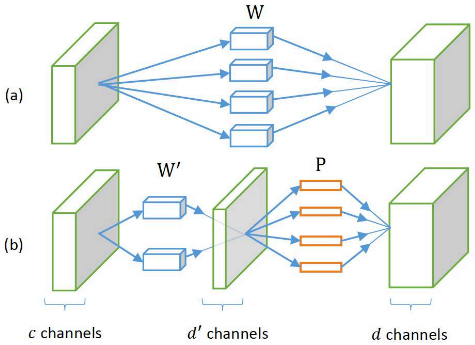
Low Rank Approximation Overview

Reading List

- Xiangyu Zhang et al. (2015). "Efficient and accurate approximations of nonlinear convolutional networks". In: *Proc. CVPR*, pp. 1984–1992
- Hao Zhou, Jose M Alvarez, and Fatih Porikli (2016). "Less is more: Towards compact cnns". In: *Proc. ECCV*, pp. 662–677
- Yihui He, Xiangyu Zhang, and Jian Sun (2017). "Channel Pruning for Accelerating Very Deep Neural Networks". In: *Proc. ICCV*
- Xiyu Yu et al. (2017). "On compressing deep models by low rank and sparse decomposition". In: *Proc. CVPR*, pp. 7370–7379

Low Rank Approximation for Conv

- Layer responses lie in a low-rank subspace
- Decompose a convolutional layer with d filters with filter size $k \times k \times c$ to
  - A layer with d' filters ($k \times k \times c$)
  - A layer with d filter ($1 \times 1 \times d'$)



Zhang et al Efficient and Accurate Approximations of Nonlinear Convolutional Networks CVPR'15

Low Rank Approximation for Conv

speedup	rank sel.	Conv1	Conv2	Conv3	Conv4	Conv5	Conv6	Conv7	err. ↑ %
2×	no	32	110	199	219	219	219	219	1.18
2×	**yes**	32	83	182	211	239	237	253	**0.93**
2.4×	no	32	96	174	191	191	191	191	1.77
2.4×	**yes**	32	74	162	187	207	205	219	**1.35**
3×	no	32	77	139	153	153	153	153	2.56
3×	**yes**	32	62	138	149	166	162	167	**2.34**
4×	no	32	57	104	115	115	115	115	4.32
4×	**yes**	32	50	112	114	122	117	119	**4.20**
5×	no	32	46	83	92	92	92	92	6.53
5×	**yes**	32	41	94	93	98	92	90	**6.47**

Zhang et al Efficient and Accurate Approximations of Nonlinear Convolutional Networks CVPR'15

Low Rank Approximation for FC

Build a mapping from row / column indices of matrix $\boldsymbol{W} = [W(x, y)]$ to vectors $\boldsymbol{i}$ and $\boldsymbol{j}$: $x \leftrightarrow \boldsymbol{i} = (i_1, \ldots, i_d)$ and $y \leftrightarrow \boldsymbol{j} = (j_1, \ldots, j_d)$.

TT-format for matrix $\boldsymbol{W}$:

$$\boldsymbol{W}(i_1, \ldots, i_d; j_1, \ldots, j_d) = \boldsymbol{W}(x(\boldsymbol{i}), y(\boldsymbol{j})) = \underbrace{\boldsymbol{G}_1[i_1, j_1]}_{1\times r}\underbrace{\boldsymbol{G}_2[i_2, j_2]}_{r\times r}\ldots\underbrace{\boldsymbol{G}_d[i_d, j_d]}_{r\times 1}$$

Type	1 im. time (ms)	100 im. time (ms)
CPU fully-connected layer	16.1	97.2
CPU TT-layer	1.2	94.7
GPU fully-connected layer	2.7	33
GPU TT-layer	1.9	12.9

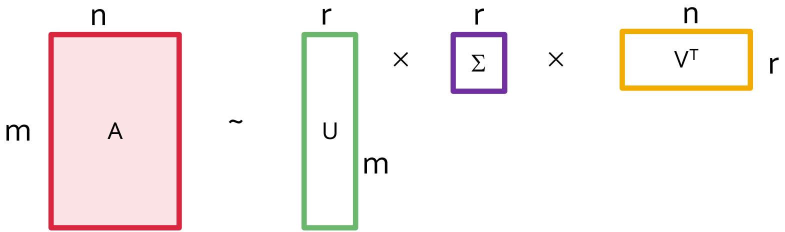
Novikov et al Tensorizing Neural Networks, NIPS'15

Singular Value Decomposition

SVD: Singular Value Decomposition

Reducing Matrix Dimension

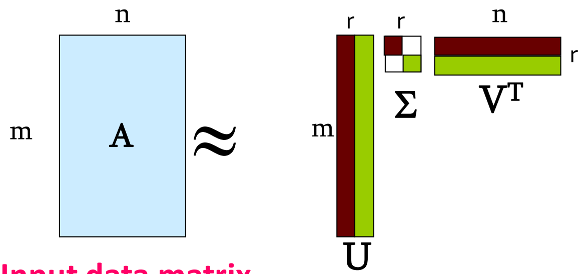
- **Gives a decomposition of any matrix into a product of three matrices:**



- There are strong constraints on the form of each of these matrices
  - Results in a unique decomposition
- From this decomposition, you can choose any number r of intermediate concepts (latent factors) in a way that minimizes the reconstruction error

SVD – Definition

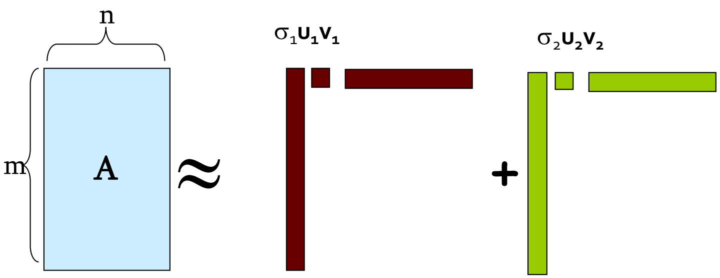
$$\mathbf{A} \approx \mathbf{U}\Sigma\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^\mathrm{T}$$



- **A**: **Input data matrix**
  - *m* x *n* matrix (e.g., *m* documents, *n* terms)
- **U**: **Left singular vectors**
  - *m* x *r* matrix (*m* documents, *r* concepts)
- **Σ**: **Singular values**
  - *r* x *r* diagonal matrix (strength of each 'concept') (*r* : rank of the matrix **A**)
- **V**: **Right singular vectors**
  - *n* x *r* matrix (*n* terms, *r* concepts)

SVD

$$\mathbf{A} \approx \mathbf{U}\Sigma\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^{\mathrm{T}}$$



σ_i … scalar

u_i … vector

v_i … vector

If we set $\sigma_2 = 0$, then the green columns may as well not exist.

SVD – Properties

It is **always** possible to decompose a real matrix $\boldsymbol{A}$ into $\boldsymbol{A} = \boldsymbol{U}\, \Sigma\, \boldsymbol{V}^{\mathsf{T}}$, where

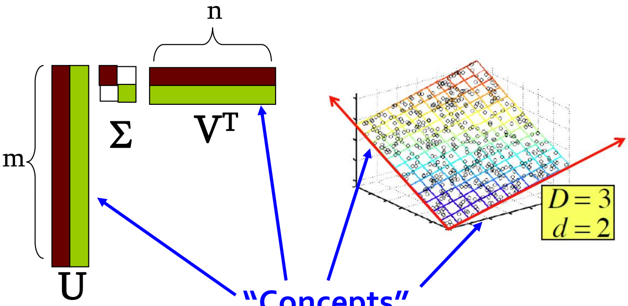
- $\boldsymbol{U}, \Sigma, \boldsymbol{V}$: unique
- $\boldsymbol{U}, \boldsymbol{V}$: column orthonormal
  - $\boldsymbol{U}^T \boldsymbol{U} = \boldsymbol{I}$; $\boldsymbol{V}^T \boldsymbol{V} = \boldsymbol{I}$ ($\boldsymbol{I}$: identity matrix)
  - (Columns are orthogonal unit vectors)
- Σ: diagonal
  - Entries (**singular values**) are non-negative, and sorted in decreasing order ($\boldsymbol{\sigma_1 \geq \sigma_2 \geq \ldots \geq 0}$)

Nice proof of uniqueness: https://www.cs.cornell.edu/courses/cs322/2008sp/stuff/TrefethenBau_Lec4_SVD.pdf

SVD – Example: Users-to-Movies

- **Consider a matrix. What does SVD do?**

$$
\begin{array}{c}
 \\
\text{SciFi} \\
 \\
\text{Romance}
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
\text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie}
\end{array} \\
\left[\begin{array}{ccccc}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{array}\right]
\end{array}
=
$$



Ratings matrix where each column corresponds to a movie and each row to a user. First 4 users prefer SciFi, while others prefer Romance.

"Concepts"
AKA Latent dimensions
AKA Latent factors

SVD – Example: Users-to-Movies

- $A = U\,\Sigma\,V^T$ - example: **Users to Movies**

$$
\begin{array}{c}
\\
\text{SciFi} \\
\\
\text{Romance}
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example: Users to Movies

$$
\begin{array}{cc}
 & \begin{array}{ccccc}\text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie}\end{array} \\
\begin{array}{c}\text{SciFi} \\ \\ \\ \\ \text{Romance} \\ \\ \\ \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{ccc}\text{SciFi-concept} & \text{Romance-concept} & \end{array} \\
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\end{array}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example:

U is "user-to-concept" factor matrix

SciFi-concept

Romance-concept

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \text{SciFi} \\ \\ \\ \\ \text{Romance} \\ \\ \\ \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ **- example:**

SciFi-concept

"strength" of the SciFi-concept

$$
\begin{array}{c}
\\ \text{SciFi} \\ \\ \\ \\ \text{Romance} \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example:

V is "movie-to-concept" factor matrix

SciFi-concept

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \text{SciFi} \\ \\ \\ \\ \text{Romance} \\ \\ \\ \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SciFi-concept

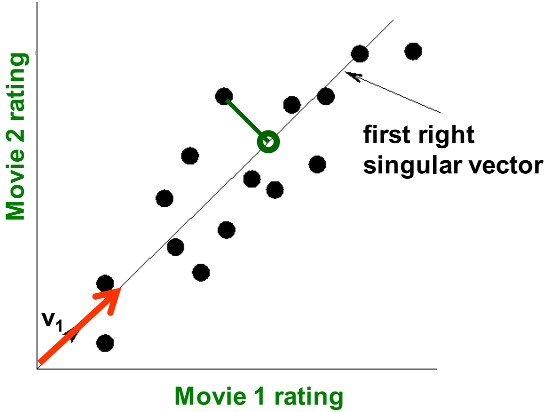
SVD – Interpretation #1

Movies, **users** and **concepts**:

- ***U***: user-to-concept matrix
- ***V***: movie-to-concept matrix
- Σ: its diagonal elements:
  'strength' of each concept

Dimensionality Reduction with SVD

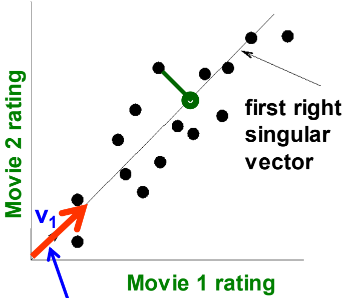
SVD – Dimensionality Reduction



- Instead of using two coordinates (x, y) to describe point positions, let's use only one coordinate
- Point's position is its location along vector v_1

SVD – Dimensionality Reduction

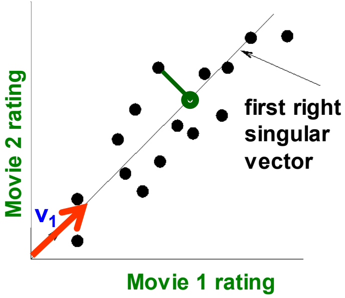
- **$A = U\,\Sigma\,V^T$ - example:**
  - **U**: "user-to-concept" matrix
  - **V**: "movie-to-concept" matrix



$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0\\
3 & 3 & 3 & 0 & 0\\
4 & 4 & 4 & 0 & 0\\
5 & 5 & 5 & 0 & 0\\
0 & 2 & 0 & 4 & 4\\
0 & 0 & 0 & 5 & 5\\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01\\
\mathbf{0.41} & 0.07 & -0.03\\
\mathbf{0.55} & 0.09 & -0.04\\
\mathbf{0.68} & 0.11 & -0.05\\
0.15 & \mathbf{-0.59} & \mathbf{0.65}\\
0.07 & \mathbf{-0.73} & \mathbf{-0.67}\\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0\\
0 & \mathbf{9.5} & 0\\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\boxed{\mathbf{0.56} \quad \mathbf{0.59} \quad \mathbf{0.56} \quad 0.09 \quad 0.09}\\
0.12 \quad -0.02 \quad 0.12 \quad \mathbf{-0.69} \quad \mathbf{-0.69}\\
0.40 \quad \mathbf{-0.80} \quad 0.40 \quad 0.09 \quad 0.09
\end{bmatrix}
$$

SVD – Dimensionality Reduction

- $A = U \Sigma V^T$ - example:



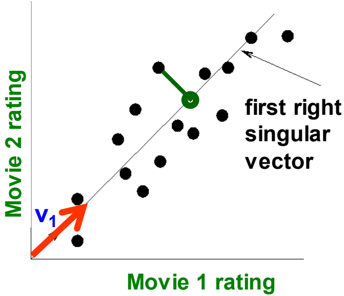
variance ('spread') on the v_1 axis

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SVD – Dimensionality Reduction

$A = U \Sigma V^T$ - example:

- $U \Sigma$: Gives the coordinates of the points in the projection axis



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Projection of users on the "Sci-Fi" axis $U \Sigma$:

$$\begin{bmatrix} \boxed{\begin{matrix} 1.61 \\ 5.08 \\ 6.82 \\ 8.43 \\ 1.86 \\ 0.86 \\ 0.86 \end{matrix}} & \begin{matrix} 0.19 \\ 0.66 \\ 0.85 \\ 1.04 \\ -5.60 \\ -6.93 \\ -2.75 \end{matrix} & \begin{matrix} -0.01 \\ -0.03 \\ -0.05 \\ -0.06 \\ 0.84 \\ -0.87 \\ 0.41 \end{matrix} \end{bmatrix}$$

SVD – Interpretation #2

More details

- **Q:** How is dim. reduction done?

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & 1 & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SVD – Interpretation #2

More details

- Q: How exactly is dim. reduction done?
- A: Set smallest singular values to zero

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & 1 & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \cancel{\mathbf{1.3}} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SVD – Interpretation #2

More details

- **Q:** **How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \cancel{\mathbf{1.3}} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SVD – Interpretation #2

This is Rank 2 approximation to A. We could also do Rank 1 approx. The larger the rank the more accurate the approximation.

More details

- Q: **How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 & \cancel{-0.01} \\
\mathbf{0.41} & 0.07 & \cancel{-0.03} \\
\mathbf{0.55} & 0.09 & \cancel{-0.04} \\
\mathbf{0.68} & 0.11 & \cancel{-0.05} \\
0.15 & \mathbf{-0.59} & \cancel{\mathbf{0.65}} \\
0.07 & \mathbf{-0.73} & \cancel{\mathbf{-0.67}} \\
0.07 & \mathbf{-0.29} & \cancel{\mathbf{0.32}}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \cancel{\mathbf{1.3}}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
\cancel{0.40} & \cancel{\mathbf{-0.80}} & \cancel{0.40} & \cancel{0.09} & \cancel{0.09}
\end{bmatrix}
$$

SVD – Interpretation #2

This is Rank 2 approximation to A. We could also do Rank 1 approx. The larger the rank the more accurate the approximation.

More details

- Q: **How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.13} & 0.02 \\ \mathbf{0.41} & 0.07 \\ \mathbf{0.55} & 0.09 \\ \mathbf{0.68} & 0.11 \\ 0.15 & \mathbf{-0.59} \\ 0.07 & \mathbf{-0.73} \\ 0.07 & \mathbf{-0.29} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 \\ 0 & \mathbf{9.5} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \end{bmatrix}$$

SVD – Interpretation #2

This is Rank 2 approximation to A. We could also do Rank 1 approx. The larger the rank the more accurate the approximation

More details

- **Q: How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} 0.92 & 0.95 & 0.92 & 0.01 & 0.01 \\ 2.91 & 3.01 & 2.91 & -0.01 & -0.01 \\ 3.90 & 4.04 & 3.90 & 0.01 & 0.01 \\ 4.82 & 5.00 & 4.82 & 0.03 & 0.03 \\ 0.70 & 0.53 & 0.70 & 4.11 & 4.11 \\ -0.69 & 1.34 & -0.69 & 4.78 & 4.78 \\ 0.32 & 0.23 & 0.32 & 2.01 & 2.01 \end{bmatrix}$$

Reconstructed data matrix B

Reconstruction Error is quantified by the Frobenius norm:

$$\|M\|_F = \sqrt{\Sigma_{ij} M_{ij}^2}$$

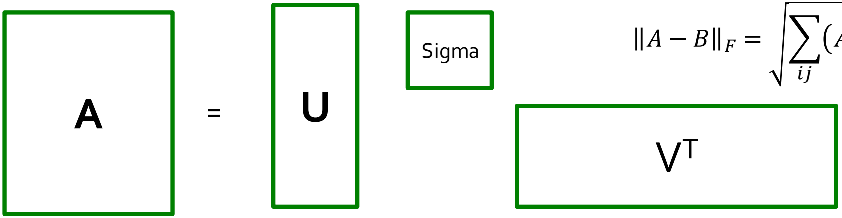
$$\|A\text{-}B\|_F = \sqrt{\Sigma_{ij} (A_{ij}-B_{ij})^2}$$

is “small”

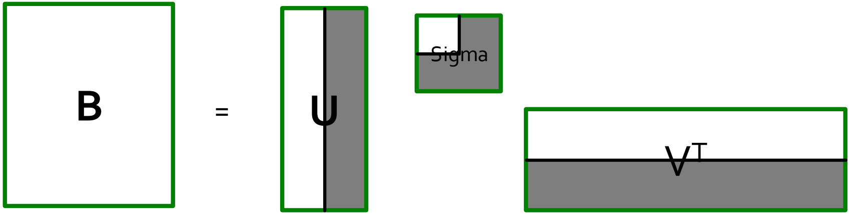
SVD – Best Low Rank Approx.

- **Fact: SVD gives 'best' axis to project on:**
  - '**best**' = minimizing the sum of reconstruction errors

$$\|A - B\|_F = \sqrt{\sum_{ij} (A_{ij} - B_{ij})^2}$$



B is best approximation of A:



SVD – Conclusions so far

- **SVD: A= U Σ V^T: unique**
  - **U**: user-to-concept factors
  - **V**: movie-to-concept factors
  - Σ : strength of each concept
- **Q: So what's a good value for r (# of latent factors)?**
- Let the *energy* of a set of singular values be the sum of their squares.
- Pick r so the retained singular values have at least 90% of the total energy.
- **Back to our example:**
  - With singular values 12.4, 9.5, and 1.3, total energy = 245.7
  - If we drop 1.3, whose square is only 1.7, we are left with energy 244, or over 99% of the total

How to Compute SVD

Finding Eigenpairs

- **How do we actually compute SVD?**
- First we need a method for finding the **principal eigenvalue** (the largest one) and the corresponding **eigenvector** of a symmetric matrix
  - M is *symmetric* if $m_{ij} = m_{ji}$ for all i and j
- **Method:**
  - Start with any "guess eigenvector" $\boldsymbol{x}_0$
  - Construct $\boldsymbol{x}_{k+1} = \frac{M\boldsymbol{x}_k}{||M\boldsymbol{x}_k||}$ for $k = 0, 1, \ldots$
    - || … || denotes the Frobenius norm
  - Stop when consecutive $\boldsymbol{x}_k$ show little change

Example: Iterative Eigenvector

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \mathbf{x}_0 = \begin{matrix} 1 \\ 1 \end{matrix}$$

$$\frac{M\mathbf{x}_0}{||M\mathbf{x}_0||} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} / \sqrt{34} = \begin{bmatrix} 0.51 \\ 0.86 \end{bmatrix} = \mathbf{x}_1$$

$$\frac{M\mathbf{x}_1}{||M\mathbf{x}_1||} = \begin{bmatrix} 2.23 \\ 3.60 \end{bmatrix} / \sqrt{17.93} = \begin{bmatrix} 0.53 \\ 0.85 \end{bmatrix} = \mathbf{x}_2$$

…..

Finding the Principal Eigenvalue

- Once you have the principal eigenvector $\boldsymbol{x}$, you find its eigenvalue λ by $\lambda = \boldsymbol{x}^T M \boldsymbol{x}$.
  - In proof: We know $\boldsymbol{x}\lambda = M\boldsymbol{x}$ if λ is the eigenvalue; multiply both sides by $\boldsymbol{x}^T$ on the left.
  - Since $\boldsymbol{x}^T\boldsymbol{x} = 1$ we have $\lambda = \boldsymbol{x}^T M \boldsymbol{x}$

- **Example:** If we take $\mathbf{x}^T = [0.53, 0.85]$, then

$$\lambda = [0.53 \; 0.85] \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0.53 \\ 0.85 \end{bmatrix} = 4.25$$

Finding More Eigenpairs

- Eliminate the portion of the matrix M that can be generated by the first eigenpair, λ and $\boldsymbol{x}$:

$$M^* := M - \lambda\, x\, x^T$$

- Recursively find the principal eigenpair for M^*, eliminate the effect of that pair, and so on

- **Example:**

$$M^* = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} - 4.25 \begin{bmatrix} 0.53 \\ 0.85 \end{bmatrix} [0.53\ 0.85] = \begin{bmatrix} -0.19 & 0.09 \\ 0.09 & 0.07 \end{bmatrix}$$

How to Compute the SVD

- **Start by supposing** $\boldsymbol{A = U\Sigma V^T}$
- $A^T = (U\Sigma V^T)^T = (V^T)^T\Sigma^T U^T = V\Sigma U^T$
  - **Why?** (1) Rule for transpose of a product; (2) the transpose of the transpose and the transpose of a diagonal matrix are both the identity functions
- $\boldsymbol{A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T}$
  - **Why?** U is orthonormal, so $U^T U$ is an identity matrix
  - Also note that Σ^2 is a diagonal matrix whose i-th element is the square of the i-th element of Σ
- $\boldsymbol{A^T A V = V\Sigma^2 V^T V = V\Sigma^2}$
  - **Why?** V is also orthonormal

Computing the SVD –(2)

- Starting with $(\boldsymbol{A}^T\boldsymbol{A})\boldsymbol{V} = \boldsymbol{V}\boldsymbol{\Sigma}^2$
  - **Note** that therefore the i-th column of V is an eigenvector of A^TA, and its eigenvalue is the i-th element of Σ^2
- Thus, we can find V and Σ by finding the eigenpairs for A^TA
  - Once we have the eigenvalues in Σ^2, we can find the singular values by taking the square root of these eigenvalues
- Symmetric argument, AA^T gives us U

SVD – Complexity

- **To compute the full SVD using specialized methods:**
  - **$O(nm^2)$** or **$O(n^2m)$** (whichever is less)
- **But:**
  - Less work, if we just want singular values
  - or if we want the first *k* singular vectors
  - or if the matrix is sparse
- **Implemented in** linear algebra packages like
  - LINPACK, Matlab, SPlus, Mathematica ...

Convolutional Neural Networks With Lowrank Regularization

Contribution

- A new algorithm for computing the low-rank tensor decomposition
- A new method for training low-rank constrained CNNs from scratch
- Evaluation on large networks

Pretrained CNN Approximation

- Convolution Calculation

$$\mathcal{F}_n(x,y)=\sum_{c=1}^{C}\sum_{x'=1}^{X}\sum_{y'=1}^{Y}\mathcal{Z}^c\left(x',y'\right)\mathcal{W}_n^c\left(x-x',y-y'\right)$$

- $\mathcal{W}_n\in\mathbb{R}^{d\times d\times C}$ to represent the n -th filter.$\mathcal{Z}\in\mathbb{R}^{X\times Y\times U}$ be the input feature map.
- An approximation of W

$$\tilde{\mathcal{W}}_n^c=\sum_{k=1}^{K}\mathcal{H}_n^k\left(\mathcal{V}_k^c\right)^T$$

  where K is a hyper-parameter controlling the rank, $\mathcal{H}\in\mathbb{R}^{N\times 1\times d\times R}$ is the horizontal filter, $\mathcal{V}\in\mathbb{R}^{K\times d\times 1\times C}$ is the vertical filter (Notes: $\mathcal{H}_k^c$ and $\mathcal{V}_k^c$ are both vectors in $\mathbb{R}^d$). Both $\mathcal{H}$ and $\mathcal{V}$ are learnable parameters.
- Then the convolution becomes

$$\tilde{W}_n*\mathcal{Z}=\sum_{c=1}^{C}\sum_{k=1}^{K}\mathcal{H}_n^k\left(\mathcal{V}_k^c\right)^T*\mathcal{Z}^c=\sum_{k=1}^{K}\mathcal{H}_n^k*\left(\sum_{c=1}^{C}\mathcal{V}_k^c*\mathcal{Z}^c\right)$$

Complexity Analysis

- Standard Convolution Complexity: $O(d^2NCXY)$ operations
- Approximation Scheme Complexity
  The computational cost associated with the vertical filters is $O(dKCXY)$ and with horizational fileters is $O(dNKXY)$, a total computational cost is $O(dK(N + C)XY)$
- If $K < \frac{dNC}{N+C}$, acceleration can be achieved

Approximate Parameters H and V

- Minimizing the objective function

$$E_1(\mathcal{H},\mathcal{V}) := \sum_{n,c} \left\| \mathcal{W}_n^c - \sum_{k=1}^K \mathcal{H}_n^k \left(\mathcal{V}_k^c\right)^T \right\|_F^2$$

- Theorem: Define the following bijection that maps a tensor to a matrix $\mathcal{T} : \mathbb{R}^{C\times d\times d\times N} \mapsto \mathbb{R}^{Cd\times dN}$, tensor element (i_1, i_2, i_3, i_4) maps to (j_1, j_2) , where

$$j_1 = (i_1 - 1)\, d + i_2, \quad j_2 = (i_4 - 1)\, d + i_3$$

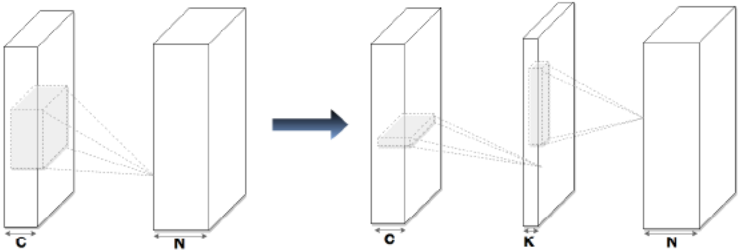
Define $W := \mathcal{T}[\mathcal{W}]$. Let $W = UDQ^T$ be the singular Value Decomposition (SVD) of W. Let

$$\hat{\mathcal{V}}_k^c(j) = U_{(c-1)d+j,k}\sqrt{D_{k,k}}$$
$$\hat{\mathcal{H}}_n^k(j) = Q_{(n-1)d+j,k}\sqrt{D_{k,k}}$$

then $(\hat{\mathcal{H}}, \hat{\mathcal{V}})$ is a solution to minimizing the object function

Singular Value Decomposition

- The proposed parametrization for low-rank regularization.



Left: The original convolutional layer. Right: low-rank constraint convolutional layer with rank-K.

Training Low-rank Constrained CNN From Scratch

- The effect of SVD Decomposition
  Each convolutional layer is parameterized as the composition of two convolutional layers,
- Exploding and vanishing gradients expecially for large networks
- Batch Normalition can handle this problem
  (Recall the theory of Batch Normalization)

Read the paper[1] if you want to learn the specific details of the algorithm

CONVOLUTIONAL NEURAL NETWORKS WITH LOW-RANK REGULARIZATION

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[1]Cheng Tai et al. (2016). "Convolutional neural networks with low-rank regularization". In: *Proc. ICLR*.

Tucker Decomposition

CP-decompositionTucker-decomposition

(a) height=2.6cm (b) height=2.6cm

Compression of Deep Convolutional Neural Networks for Fast and Low Power Mobile Applications

Contribution

- Propose a one-shot whole network compression scheme which consists of simple three steps: (1) rank selection, (2) low-rank tensor decomposition, and (3) fine-tuning.
- Tucker decomposition (Tucker, 1966) with the rank determined by a global analytic solution of variational Bayesian matrix factorization is applied on each kernel tensor.

Kernel Tensor Approximation

- Convolution Calculation

$$\mathcal{Y}_{h',w',t} = \sum_{i=1}^{D} \sum_{j=1}^{D} \sum_{s=1}^{S} \mathcal{K}_{i,j,s,t} \mathcal{X}_{h_i,w_j,s}$$

$$h_i = (h' - 1)\,\Delta + i - P \text{ and } w_j = (w' - 1)\,\Delta + j - P$$

where $\mathcal{K}$ is a 4-way kernel tensor of size $D \times D \times S \times T$, δ is stride, and P is zero-padding size

- Tucker Decomposition:The rank-$(R_1; R_2; R_3; R_4)$ Tucker decomposition of 4-way kernel tensor K has the form:

$$\mathcal{K}_{i,j,s,t} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \sum_{r_9=1}^{R_3} \sum_{r_4=1}^{R_4} \mathcal{C}'_{r_1,r_2,r_3,r_4} U^{(1)}_{i,r_1} U^{(2)}_{j,r_2} U^{(3)}_{s,r_3} U^{(4)}_{t,r_4}$$

where $\mathcal{C}'$ is a core tensor of size $R_1 \times R_2 \times R_3 \times R_4$ and $U^{(1)}, U^{(2)}, U^{(3)}$, and $U^{(4)}$ are factor matrices of sizes $D \times R_1, D \times R_2, S \times R_3$, and $T \times R_4$, respectively.

Tucker Decomposition

- Every mode does not have to be decomposed(e.g. For example, we do not decompose mode-1 and mode-2 which are associated with spatial dimensions because they are already quite small).
- Under this variant called Tucker-2 decomposition, the kernel tensor is decomposed to:

$$\mathcal{K}_{i,j,s,t} = \sum_{r_0=1}^{R_0} \sum_{r_4=1}^{R_4} \mathcal{C}_{i,j,r_3,r_4} U^{(3)}_{s,r_0} U^{(4)}_{t,r_4}$$

  where $\mathcal{C}$ is a core tensor of size $D \times D \times R_3 \times R_4$

- With the approximation of kernel, the convolution is as following:

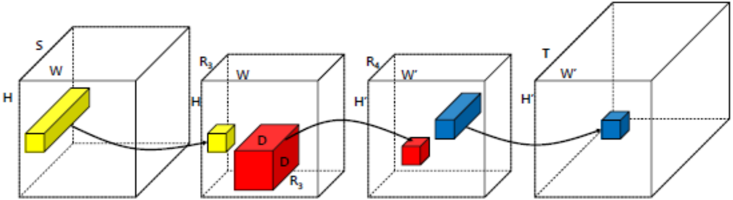
$$\mathcal{Z}_{h,w,r_3} = \sum_{s=1}^{S} U^{(3)}_{s,r_3} \mathcal{X}_{h,w,s}$$

$$\mathcal{Z}'_{h',w',r_4} = \sum_{i=1}^{D} \sum_{j=1}^{D} \sum_{r_0=1}^{R_0} \mathcal{C}_{i,j,r_3,r_4} \mathcal{Z}_{h_t,w_j,r_9}$$

R_4

Tucker Decomposition

- Tucker-2 decompositions for speeding-up a convolution



- Complexity Analysis

$$M = \frac{D^2 ST}{SR_3 + D^2 R_3 R_4 + TR_4} \text{ and } E = \frac{D^2 STH'W'}{SR_3 HW + D^2 R_3 R_4 H'W' + TR_4 H'W'}$$

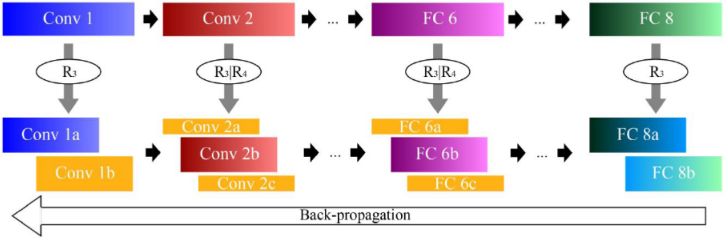
M represents the compression ratio, E represents the speed-up ratio

Rank Selection With Global Analytic VBMF

- Motivation: The rank-($R_3; R_4$) control the trade-off between performance (memory, speed, energy) improvement and accuracy loss.
- Method: variational Bayesian atrix factorization[2]
- Advantages: VBMF can automatically find noise variance, rank and even provide theoretical condition for perfect rank recovery

[2]Shinichi Nakajima et al. (2013). "Global analytic solution of fully-observed variational Bayesian matrix factorization". In: *Journal of Machine Learning Research* 14.Jan, pp. 1–37.

- One-shot whole network compression scheme



Three parts: (1) rank selection with VBMF; (2) Tucker decomposition on kernel tensor; (3) fine-tuning of entire network.

- Notes:Tucker-2 decomposition is applied from the second convolutional layer to the first fully connected layers, and Tucker-1 decomposition to the other layers.

Tucker Decomposition

Read the paper[3] if you want to learn the specific details of the algorithm

COMPRESSION OF DEEP CONVOLUTIONAL NEURAL NETWORKS FOR FAST AND LOW POWER MOBILE APPLICATIONS

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[3]Yong-Deok Kim et al. (2016). "Compression of deep convolutional neural networks for fast and low power mobile applications". In: *Proc. ICLR*.

CP-Decomposition

Advantages

- Ease of the decomposition implementation
- Ease of the CNN implementation
- Ease of fine-tuning
- Efficiency

Speeding-up Convolutional Neural Networks Using Fine-tuend CP-Decomposition

Method Overview

- Take a convolutional layer and decompose its kernel using CP-decomposition
- Fine-tune the entire network using backpropagation.

Principle

- A low-rank decomposition of a matrix A of size $n \times m$ with rank R is given by

$$A(i,j) = \sum_{r=1}^{R} A_1(i,r)A_2(j,r), \quad i = \overline{1,n}, \quad j = \overline{1,m}$$

- For a d-dimensional array A of size $n_1 \times \cdots \times n_d$ a CP-decomposition has the following form

$$A\left(i_1, \ldots, i_d\right) = \sum_{r=1}^{R} A_1\left(i_1, r\right) \ldots A_d\left(i_d, r\right)$$

where the minimal possible R is called canonical rank.

- Profit we need to store only $\left(n_1 + \cdots + n_d\right) R$ elements instead of the whole tensor with $n_1 \ldots n_d$ elements.

Notes:

- There is no finite algorithm for determining canonical rank of a tensor when $d > 2$
- Non-linear least squares (NLS) method minimizes the L2-norm of the approximation residual (for a user-defined fixed R) using Gauss-Newton optimization.

Kernel Tensor Approximation

- Convolution Calculation

$$V(x, y, t) = \sum_{i=x-\delta}^{x+\delta} \sum_{j=y-\delta}^{y+\delta} \sum_{s=1}^{S} K(i-x+\delta, j-y+\delta, s, t) U(i, j, s)$$

- $K(\cdot, \cdot, \cdot, \cdot)$ is a 4D kernel tensor of size $d \times d \times S \times T$ d is the spatial dimensions, S is input channels, T is output channels, while δ denotes "half-width" $(d-1)/2$
- Kernel Approximation

$$K(i, j, s, t) = \sum_{r=1}^{R} K^x(i-x+\delta, r) K^y(j-y+\delta, r) K^s(s, r) K^t(t, r)$$

- where $K^x(\cdot, \cdot), K^y(\cdot, \cdot), K^s(\cdot, \cdot), K^t(\cdot, \cdot)$ are the four components of the composition representing 2D tensors (matrices) of sizes $d \times R, d \times R, S \times R$, and $T \times R$ respectively.

Convolution Approximation

- Substitue the Kernel Approx to Conv

$$V(x,y,t) = \sum_{r=1}^{R} K^t(t,r)\left(\sum_{i=x-\delta}^{x+\delta} K^x(i-x+\delta,r)\left(\sum_{j=y-\delta}^{y+\delta} K^y(j-y+\delta,r)\left(\sum_{s=1}^{S} K^s(s,r)U(i,j,s)\right)\right)\right)$$

- Step by Step Calculation

$$U^s(i,j,r) = \sum_{s=1}^{S} K^s(s,r)U(i,j,s)$$

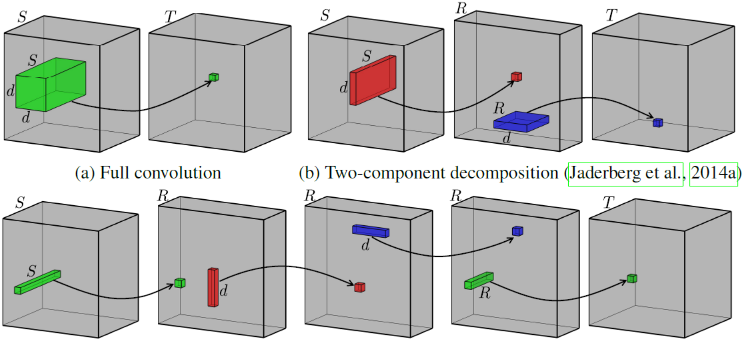
$$U^{sy}(i,y,r) = \sum_{j=y-\delta}^{y+\delta} K^y(j-y+\delta,r)U^s(i,j,r)$$

$$U^{\mathrm{syx}}(x,y,r) = \sum_{i=x-\delta}^{x+\delta} K^x(i-x+\delta,r)U^{sy}(i,y,r)$$

$$V(x,y,t) = \sum_{r=1}^{R} K^t(t,r)U^{\mathrm{syx}}(x,y,r)$$

CP-Decomposition

- Complexity Comparison



(a) Full convolution (b) Two-component decomposition (Jaderberg et al., 2014a)

(c) CP-decomposition

CP-Decomposition

Read the paper[4] if you want to learn the specific details of the algorithm

SPEEDING-UP CONVOLUTIONAL NEURAL NETWORKS USING FINE-TUNED CP-DECOMPOSITION

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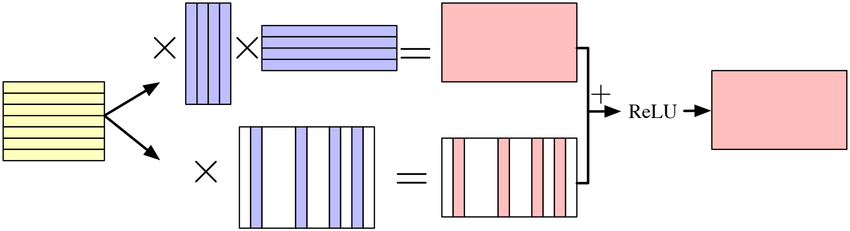
[4]Vadim Lebedev et al. (2015). “Speeding-up convolutional neural networks using fine-tuned CP-decomposition”. In: *Proc. ICLR*.

Overview



- Simultaneous low-rank approximation and network sparsification;
- Non-linearity is taken into account.
- Acceleration is achieved with structured sparsity.

Formulation

Given a pre-trained network, the goal is to minimize the reconstruction error of the response in each layer after activation, using sparse component and low-rank component.

$$
\begin{aligned}
\min_{\boldsymbol{A},\boldsymbol{B}} \; & \sum_{i=1}^{N} \|\boldsymbol{Y}_i - r((\boldsymbol{A}+\boldsymbol{B})\boldsymbol{X}_i)\|_F \,, \\
\text{s.t.} \; & \|\boldsymbol{A}\|_0 \leq S, \\
& \operatorname{rank}(\boldsymbol{B}) \leq L.
\end{aligned}
$$

- $\boldsymbol{X}$: input feature map
- $\boldsymbol{Y}$: output feature map

Not easy to solve: l_0 minimization and rank minimization are NP-hard.

$$\min_{\boldsymbol{A},\boldsymbol{B}} \sum_{i=1}^{N} \|\boldsymbol{Y}_i - r((\boldsymbol{A}+\boldsymbol{B})\boldsymbol{X}_i)\|_F^2 + \lambda_1 \|\boldsymbol{A}\|_{2,1} + \lambda_2 \|\boldsymbol{B}\|_*$$

- The l_0 constraint is relaxed by $l_{2,1}$ norm such that the zero elements in $\boldsymbol{A}$ appear column-wise;
- The rank constraint on $\boldsymbol{B}$ is relaxed by nuclear norm of $\boldsymbol{B}$, which is the sum of the singular values;
- Apply alternating direction method of multipliers (ADMM) to solve it;

Reformulating the problem with an auxiliary variable $\boldsymbol{M}$,

$$
\begin{aligned}
&\min_{\boldsymbol{A},\boldsymbol{B},\boldsymbol{M}} \sum_{i=1}^{N} \|\boldsymbol{Y}_i - r(\boldsymbol{M}\boldsymbol{X}_i)\|_F^2 + \lambda_1 \|\boldsymbol{A}\|_{2,1} + \lambda_2 \|\boldsymbol{B}\|_* , \\
&\text{s.t.} \;\; \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{M}.
\end{aligned}
$$

Then the augmented Lagrangian function is

$$
\begin{aligned}
& L_t(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{M}, \boldsymbol{\Lambda}) \\
= & \sum_{i=1}^{N} \|\boldsymbol{Y}_i - r(\boldsymbol{M}\boldsymbol{X}_i)\|_F^2 + \lambda_1 \|\boldsymbol{A}\|_{2,1} + \lambda_2 \|\boldsymbol{B}\|_* + \langle \boldsymbol{\Lambda}, \boldsymbol{A} + \boldsymbol{B} - \boldsymbol{M} \rangle + \frac{t}{2} \|\boldsymbol{A} + \boldsymbol{B} - \boldsymbol{M}\|_F^2 ,
\end{aligned}
$$

Iteratively solve with following rules. All of them can be solved efficiently.

$$
\begin{cases}
\boldsymbol{A}_{k+1} = \underset{\boldsymbol{A}}{\operatorname{argmin}}\ \lambda_1 \|\boldsymbol{A}\|_{2,1} + \dfrac{t}{2}\left\|\boldsymbol{A} + \boldsymbol{B}_k - \boldsymbol{M}_k + \dfrac{\boldsymbol{\Lambda}_k}{t}\right\|_F^2, \\
\boldsymbol{B}_{k+1} = \underset{\boldsymbol{B}}{\operatorname{argmin}}\ \lambda_2 \|\boldsymbol{B}\|_* + \dfrac{t}{2}\left\|\boldsymbol{B} + \boldsymbol{A}_{k+1} - \boldsymbol{M}_k + \dfrac{\boldsymbol{\Lambda}_k}{t}\right\|_F^2, \\
\boldsymbol{M}_{k+1} = \underset{\boldsymbol{M}}{\operatorname{argmin}}\ \sum_{i=1}^{N} \|\boldsymbol{Y}_i - r(\boldsymbol{M}\boldsymbol{X}_i)\|_F^2 + \langle \boldsymbol{\Lambda}_k, \boldsymbol{A}_{k+1} + \boldsymbol{B}_{k+1} - \boldsymbol{M} \rangle + \dfrac{t}{2}\|\boldsymbol{A}_{k+1} + \boldsymbol{B}_{k+1} - \boldsymbol{M}\|_F^2, \\
\boldsymbol{\Lambda}_{k+1} = \boldsymbol{\Lambda}_k + t(\boldsymbol{A}_{k+1} + \boldsymbol{B}_{k+1} - \boldsymbol{M}_{k+1}).
\end{cases}
$$

Solving $l_{2,1}$-norm

$$\min_{\boldsymbol{A}} \lambda_1 \|\boldsymbol{A}\|_{2,1} + \frac{t}{2} \left\| \boldsymbol{A} + \boldsymbol{B}_k - \boldsymbol{M}_k + \frac{\boldsymbol{\Lambda}_k}{t} \right\|_F^2$$

Closed Form Update Rule[5]

$$\boldsymbol{A}_{k+1} = \mathrm{prox}_{\frac{\lambda_1}{t}\|\cdot\|_{2,1}}(\boldsymbol{M}_k - \boldsymbol{B}_k - \frac{\boldsymbol{\Lambda}_k}{t}),$$

$$\boldsymbol{C} = \boldsymbol{M}_k - \boldsymbol{B}_k - \frac{\boldsymbol{\Lambda}_k}{t},$$

$$[\boldsymbol{A}_{k+1}]_{:,i} = \begin{cases} \dfrac{\|[\boldsymbol{C}]_{:,i}\|_2 - \frac{\lambda_1}{t}}{\|[\boldsymbol{C}]_{:,i}\|_2}[\boldsymbol{C}]_{:,i}, & \text{if } \|[\boldsymbol{C}]_{:,i}\|_2 > \dfrac{\lambda_1}{t}; \\ 0, & \text{otherwise.} \end{cases}$$

[5]G. Liu et al., *"Robust recovery of subspace structures by low-rank representation"*, *TPAMI, 2013.*

$$\min_{\boldsymbol{B}} \lambda_2 \|\boldsymbol{B}\|_* + \frac{t}{2} \left\| \boldsymbol{B} + \boldsymbol{A}_{k+1} - \boldsymbol{M}_k + \frac{\boldsymbol{\Lambda}_k}{t} \right\|_F^2$$

Closed Form Update Rule[6]

$$\boldsymbol{B}_{k+1} = \mathrm{prox}_{\frac{\lambda_2}{t}\|\cdot\|_*}(\boldsymbol{M}_k - \boldsymbol{A}_{k+1} - \frac{\boldsymbol{\Lambda}_k}{t}),$$

$$\boldsymbol{D} = \boldsymbol{M}_k - \boldsymbol{A}_{k+1} - \frac{\boldsymbol{\Lambda}_k}{t},$$

$$\boldsymbol{B}_{k+1} = \boldsymbol{U}\mathcal{D}_{\frac{\lambda_2}{t}}(\boldsymbol{\Sigma})\boldsymbol{V}, \quad \text{where } \mathcal{D}_{\frac{\lambda_2}{t}}(\boldsymbol{\Sigma}) = \mathrm{diag}(\{(\sigma_i - \frac{\lambda_2}{t})_+\}).$$

[6] J-F. Cai et al., *"A singular value thresholding algorithm for matrix completion", SIOPT, 2010.*

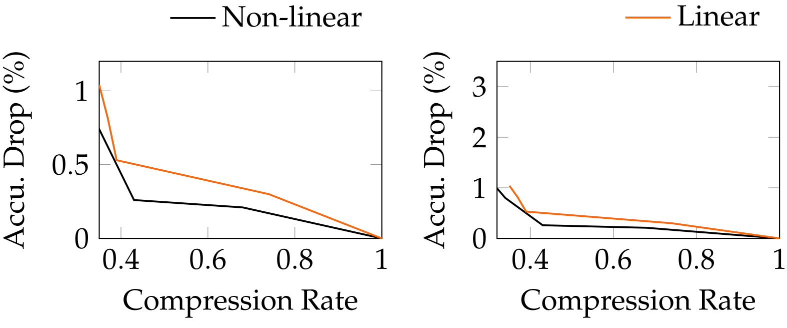
Comparison on *CIFAR-10* dataset

Model	Method	Accuracy ↓	CR	Speed-up
VGG-16	Original	0.00%	1.00	1.00
	ICLR'17[7]	**0.06%**	2.70	1.80
	Ours	0.40%	**4.44**	**2.20**
NIN	Original	0.00%	1.00	1.00
	ICLR'16[8]	1.43%	1.54	1.50
	IJCAI'18[9]	1.43%	1.45	-
	Ours	**0.41%**	**2.77**	**1.70**

[7] Hao Li et al. (2017). "Pruning filters for efficient convnets". In: *Proc. ICLR*.

[8] Cheng Tai et al. (2016). "Convolutional neural networks with low-rank regularization". In: *Proc. ICLR*.

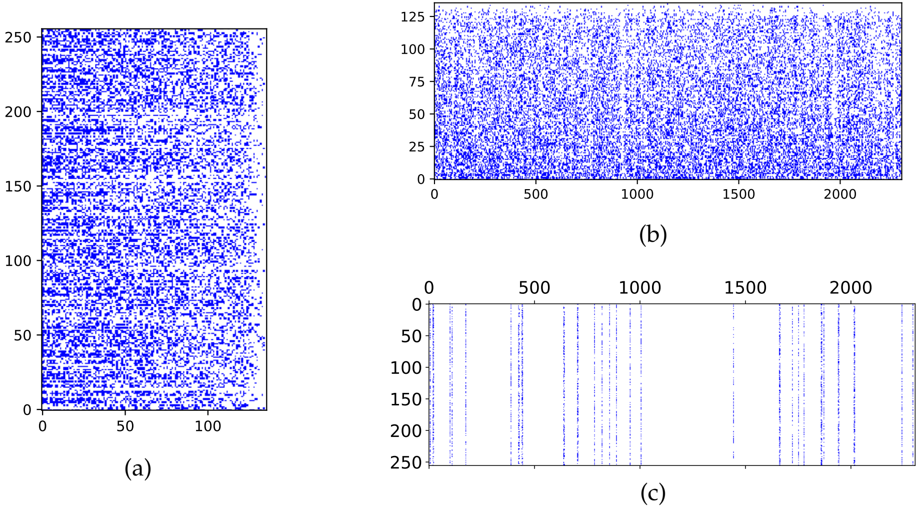
[9] Shiva Prasad Kasiviswanathan, Nina Narodytska, and Hongxia Jin (2018). "Network Approximation using Tensor Sketching". In: *Proc. IJCAI*, pp. 2319–2325.



(a) (b)

Comparison of reconstructing linear response and non-linear response: (a) layer `conv2-1`; (b) layer `conv3-1`.

Approximation Example



Approximated filters of `conv3-1`. Blue dots have non-zero values. Low-rank filter $\boldsymbol{B}$ with rank 136 is decomposed into $\boldsymbol{UV}$, both of which have rank 136. (a) Matrix $\boldsymbol{U}$; (b) Matrix $\boldsymbol{V}$. (c) Column-wise sparse filter $\boldsymbol{A}$.

Comparison on *ImageNet* dataset

Model	Method	Top-5 Accu.↓	CR	Speed-up
AlexNet	Original	0.00%	1.00	1.00
	ICLR'16[10]	**0.37%**	5.00	**1.82**
	ICLR'16[11]	1.70%	5.46	1.81
	CVPR'18[12]	1.43%	1.50	-
	Ours	1.27%	**5.56**	1.10
GoogleNet	Original	0.00%	1.00	1.00
	ICLR'16[10]	0.42%	2.84	1.20
	ICLR'16[11]	0.24%	1.28	1.23
	CVPR'18[12]	0.21%	1.50	-
	Ours	**0.00%**	**2.87**	**1.35**

[10] Cheng Tai et al. (2016). "Convolutional neural networks with low-rank regularization". In: *Proc. ICLR*.

[11] Yong-Deok Kim et al. (2016). "Compression of deep convolutional neural networks for fast and low power mobile applications". In: *Proc. ICLR*.

[12] Ruichi Yu et al. (2018). "NISP: Pruning networks using neuron importance score propagation". In: *Proc. CVPR*, pp. 9194–9203.