Proof of EPI - review

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1 PRELIMINARIES

Let $X$ be a continuous random variable with a density. Define $J(X)$ by

$$J(X) = \left. \frac{d}{ds} h(X + \sqrt{s}Z) \right|_{s \to 0^+}$$

where $Z \sim \mathcal{N}(0, 1)$ is independent of $X$.

Lemma 1. Let $X$ be a continuous random variable with a density and $Z \sim \mathcal{N}(0, 1)$ be independent of $X$. We show that $J(\cdot)$ satisfies the following:

(a) $J(X + \sqrt{s}Z) = \left. \frac{d}{ds} h(X + \sqrt{s}Z) \right|_{s \to 0^+}$, when $s > 0$.

(b) $J(aX) = \frac{1}{a^2} J(X)$.

(c) $\frac{d}{dt} h(X\sqrt{t} + \sqrt{1-t}Z) = -\frac{1}{t} J(X\sqrt{t} + \sqrt{1-t}Z) + \frac{1}{2\ln 2}.$

Proof. (a): Let $Z_1 \sim \mathcal{N}(0, 1)$ be independent of $X$ and $Z$.

$$\frac{d}{ds} h(X + \sqrt{s}Z) = \lim_{\delta \to 0} \frac{h(X + \sqrt{s + \delta}Z) - h(X + \sqrt{s}Z)}{\delta}$$

$$= \lim_{\delta \to 0} \frac{h(X + \sqrt{s}Z + \sqrt{\delta}Z_1) - h(X + \sqrt{s}Z)}{\delta}$$

$$= \left. \frac{d}{dt} h(X + \sqrt{t}Z + \sqrt{1-t}Z_1) \right|_{t=0}$$

$$= J(X + \sqrt{s}Z)$$

(b): W.l.o.g. $a > 0$ and let $u = \frac{s}{a^2}$. Observe that

$$J(aX) = \left. \frac{d}{ds} h(aX + \sqrt{s}Z) \right|_{s=0} = \left. \frac{d}{ds} (h(X + \frac{1}{a}\sqrt{s}Z) + \log_2 a) \right|_{s=0} = \frac{1}{a^2} \left. \frac{d}{du} h(X + \sqrt{u}Z) \right|_{u=0} = \frac{1}{a^2} J(X).$$

1 In fact $J(X)$ is (scaled) Fisher information. The existence of the right limit can be shown by dominated convergence.
(c): Let $s = \frac{1-t}{t}$. Note that $\frac{ds}{dt} = -\frac{1}{t^2}$. Observe that

$$
\frac{d}{dt} h(X\sqrt{t} + \sqrt{1-t} Z) = \frac{d}{dt} \left( h(X + \sqrt{\frac{1-t}{t}} Z) + \frac{1}{2t \ln 2} \right)
$$

$$
= -\frac{1}{t^2} \frac{d}{ds} h(X + \sqrt{s} Z) + \frac{1}{2t \ln 2}
$$

$$
= -\frac{1}{t^2} J(X + \sqrt{s} Z) + \frac{1}{2t \ln 2}
$$

$$
= -\frac{1}{t^2} J(X + \sqrt{\frac{1-t}{t}} Z) + \frac{1}{2t \ln 2}
$$

where (a) follows from part (a) and (b) follows from part (b).

\section{Main}

\textbf{Theorem 1.} Let $X$ and $Y$ be independent continuous random variables. Let $\lambda \in (0, 1)$. We have

$$
J(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \leq \lambda J(X) + (1-\lambda) J(Y).
$$

\textbf{Proof.} This is a consequence of data-processing inequality. Let $Z \sim \mathcal{N}(0, 1)$ be independent of $X, Y$. Let $X_t = X + \sqrt{\lambda} t Z, Y_t = X + \sqrt{(1-\lambda) t} Z$. Note that we have the following two Markov chains:

- $Z \rightarrow (X_t, Y_t) \rightarrow \sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t$,
- $X_t \rightarrow Z \rightarrow Y_t$.

Hence, by data processing inequality, we have

$$
I(Z; \sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) \leq I(Z; X_t, Y_t)
$$

$$
\leq I(Z; X_t) + I(X_t, Z; Y_t)
$$

$$
= I(Z; X_t) + I(Z; Y_t).
$$

Define

$$
g(t) = I(Z; X_t) + I(Z; Y_t) - I(Z; \sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t)).
$$

Note that $g(0) = 0$ and $g(t) \geq 0$ for $t \geq 0$. Hence $g'(0) \geq 0$.

Let $u = \lambda t$. Observe that

$$
\frac{d}{dt} I(Z; X_t) = \frac{d}{dt} h(X_t) = \frac{d}{dt} h(X + \sqrt{\lambda t} Z) = \lambda \frac{d}{du} h(X + \sqrt{u} Z) = \lambda J(X + \sqrt{u} Z) = \lambda J(X + \sqrt{\lambda t} Z),
$$

where the last-but-one step follows from part (a) of Lemma 1.

Similarly,

$$
\frac{d}{dt} I(Z; Y_t) = (1-\lambda) J(X + \sqrt{1-\lambda} t Z).
$$

2
Since $\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t = \sqrt{\lambda}X + \sqrt{1-\lambda}Y + \sqrt{t}Z$, we have from part a) of Lemma 1 that

$$\frac{d}{dt} I(Z; \sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) = \frac{d}{dt} h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y + \sqrt{t}Z) = J(\sqrt{\lambda}X + \sqrt{1-\lambda}Y + \sqrt{t}Z).$$

Hence

$$g'(0) = \lambda J(X) + (1-\lambda)J(Y) - J(\sqrt{\lambda}X + \sqrt{1-\lambda}Y),$$

and since $g'(0) \geq 0$ the theorem follows.

**Theorem 2.** Let $X$, $Y$ be continuous independent variables. Let $\lambda \in (0, 1)$. We have

$$h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda h(X) + (1-\lambda)h(Y).$$

**Proof.** For $t \in [0, 1]$, let $X_t = \sqrt{t}X + \sqrt{1-t}X_0$, $Y_t = \sqrt{t}X + \sqrt{1-t}Y_0$ where $X_0$, $Y_0 \sim N(0, 1)$ are independent of each other and of $X$, $Y$. Define

$$f(t) = h(\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) - \lambda h(X_t) - (1-\lambda)h(Y_t).$$

When $t = 0$, $X_t = X_0$, $Y_t = Y_0$ and observe that

$$f(0) = \frac{1}{2} \log 2\pi e - \lambda \frac{1}{2} \log 2\pi e - (1-\lambda) \frac{1}{2} \log 2\pi e = 0$$

We need to show that $f'(1) \geq 0$. Since $f(0) = 0$, it suffices to show $f'(t) \geq 0$ when $t \in (0, 1)$.

Observe that $\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t \sim \sqrt{t}(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) + \sqrt{1-t}Z$. Hence, using part (c) of Lemma 1 we have,

$$\frac{d}{dt} h(\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) = \frac{d}{dt} h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y + \sqrt{t}Z)$$

$$= -\frac{1}{t} f(\sqrt{\lambda}X + \sqrt{1-\lambda}Y + \sqrt{1-t}Z) + \frac{1}{2t \ln 2}$$

$$= -\frac{1}{t} f(\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) + \frac{1}{2t \ln 2}.$$ 

From part (c) of Lemma 1 we also have

$$\frac{d}{dt} h(X_t) = -\frac{1}{t} f(\sqrt{t}X + \sqrt{1-t}X_0) + \frac{1}{2t \ln 2}$$

$$\frac{d}{dt} h(Y_t) = -\frac{1}{t} f(\sqrt{t}Y + \sqrt{1-t}Y_0) + \frac{1}{2t \ln 2}.$$ 

Thus we have

$$f'(t) = \frac{d}{dt} h(\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) - \lambda \frac{d}{dt} h(X_t) - (1-\lambda) \frac{d}{dt} h(Y_t)$$

$$= -\frac{1}{t} \left( f(\sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t) - \lambda f(X_t) - (1-\lambda) f(Y_t) \right)$$

$$\geq 0$$

as desired. The last step follows from Theorem 1 as $X_t$, $Y_t$ are independent. 

□
Theorem 3. The two statements below are equivalent:

1. \(2^{2h(X+Y)} \geq 2^{2h(X)} + 2^{2h(Y)}\) holds for all continuous and independent \(X, Y\);

2. \(h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda h(X) + (1-\lambda) h(Y)\) holds for all continuous and independent \(X, Y\) and \(\lambda \in (0, 1)\).

Proof. \((1) \implies (2)\):

\[
2^{2h(\sqrt{\lambda}X + \sqrt{1-\lambda}Y)} \geq 2^{2h(X)+\log_2 \lambda} + 2^{2h(Y)+\log_2 (1-\lambda)}
\]

\[
= \lambda 2^{2h(X)} + (1-\lambda)2^{2h(Y)}
\]

\[
\geq 2^{2\lambda h(X)+2(1-\lambda)h(Y)}
\]

Last step follows from convexity of the function \(2^x\).

\((2) \implies (1)\): From (2), we have

\[
h(X + Y) \geq \lambda h\left(\frac{X}{\sqrt{\lambda}}\right) + (1-\lambda) h\left(\frac{Y}{\sqrt{1-\lambda}}\right).
\]

Let

\[
\lambda = \frac{2^{2h(X)}}{2^{2h(X)}+2^{2h(Y)}}.
\]

Then, note that

\[
h\left(\frac{X}{\sqrt{\lambda}}\right) = h(X) - \frac{1}{2} \log_2 \frac{2^{2h(X)}+2^{2h(Y)}}{2^{2h(X)}} = \frac{1}{2} \log_2 (2^{2h(X)}+2^{2h(Y)}),
\]

\[
h\left(\frac{Y}{\sqrt{1-\lambda}}\right) = h(Y) - \frac{1}{2} \log_2 \frac{2^{2h(Y)}}{2^{2h(X)}+2^{2h(Y)}} = \frac{1}{2} \log_2 (2^{2h(X)}+2^{2h(Y)}).
\]

So we have

\[
h(X + Y) \geq \frac{1}{2} \log_2 (2^{2h(X)}+2^{2h(Y)}), \text{ or equivalently } 2^{2h(X+Y)} \geq 2^{2h(X)} + 2^{2h(Y)}.
\]

\[\square\]

Entropy Power Inequality: Since the statement 2 in Theorem 3 holds by Theorem 2, we have the celebrated Entropy Power Inequality that

\[
2^{2h(X+Y)} \geq 2^{2h(X)} + 2^{2h(Y)}
\]

holds for all continuous and independent \(X, Y\).

TO DO

- Show existence of the limit in the definition of \(J(X)\).
- Derive the condition for equality in EPI.