Gap Amplification Fails Below 1/2

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Abstract

The gap amplification lemma of Dinur (ECCC TR05-46) states that the satisfiability gap of every d-regular constraint expander graph G (with self-loops) can be amplified by graph powering, as long as the satisfiability gap of G is not too large. We show that the last requirement is necessary. Namely, for infinitely many d and every t there exists an integer n and a d-regular constraint expander G on n vertices over alphabet $\{0, 1\}$ such that $\overline{\text{SAT}}(G) \geq 1/2 - O(1/\sqrt{d})$, but $\overline{\text{SAT}}(G^t) \leq 1/2$.

The main technical tool in Dinur's recent combinatorial proof of the PCP theorem [Din05] is the following gap amplification lemma:

Lemma 1 ([Din05, Lemma 3.4]). *Let $\lambda < d$, and $|\Sigma|$ be arbitrary constants. There exists a constant $\beta = \beta(\lambda, d, |\Sigma|)$ such that for every t and every d-regular constraint graph G over alphabet Σ with self-loops and $\lambda(G) < \lambda$, $\overline{\text{SAT}}(G^t) \geq \beta\sqrt{t}\min(\overline{\text{SAT}}(G), 1/t)$.*

Here $\lambda(G)$ denotes the second largest eigenvalue of the graph G, and $\overline{\text{SAT}}(G)$ denotes the *satisfiability gap* of G, namely the fraction of constraints of G that every assignment leaves unsatisfied.

A question of interest is whether the dependency on $1/t$ is necessary in the above statement. In particular, is it true that for large enough $t = t(\beta)$, say, $\overline{\text{SAT}}(G^t) \geq 2\overline{\text{SAT}}(G)$? Such a result would imply, for arbitrary $\epsilon > 0$, the NP-hardness of distinguishing whether instances of a certain type of 2-CSP are satisfiable or $1-\epsilon$ far from satisfiable,[1] thereby providing an alternative to Raz's parallel repetition theorem [Raz95] in certain applications.

This is, however, not the case. In fact, we show that for every pair of constants d and t there exists an integer n and a d-regular constraint expander G with self-loops on n vertices over alphabet $\{0, 1\}$ such that $\overline{\text{SAT}}(G) \leq 1/2 + O(1/\sqrt{d})$, but $\overline{\text{SAT}}(G^t) \geq 1/2$. We make use of the following construction:

Construction 2. *For infinitely many integers d there exist infinitely many n and a d-regular graph on n vertices G with: (1) G has girth $\frac{2}{3}\log_d n$; (2) $\lambda(G) = 2\sqrt{d-1}$; (3) every two-partition of G is violated by at least a $1/2 - 2/\sqrt{d-1}$ fraction of edges.*

Proof. The non-bipartite expanders of Lubotzky et al. [LPS88] have the desired properties. Properties (1) and (2) are explicit in [LPS88]. We derive (3) from (2). By the expander mixing lemma, for every set S of vertices of size θn,

$$\left|e(S, \overline{S}) - \theta(1-\theta)dn\right| \leq \lambda(G)\sqrt{\theta(1-\theta)}n,$$

[1] The alphabet size would depend on ϵ but not on the instance size.

where $e(S, \overline{S})$ is the number of edges crossing the cut $(S, \overline{S})$. Since $\theta(1-\theta)$ is maximized at $\theta = 1/2$, we have that

$$e(S, \overline{S}) \leq dn/4 + \sqrt{d-1}n.$$

Therefore every partition is violated by at least $dn/4 - \sqrt{d-1}n$ edges, establishing property (3). □

Take a graph G given by the construction, and add a self-loop to every vertex. Now consider the following constraint satisfaction problem on G: The alphabet is $\Sigma = \{0, 1\}$, the edge constraints are dummy (always satisfied) on loops, and inequality constraints on the other edges. By property (3) of the construction, $\overline{\text{SAT}}(G) \geq 1/2 - O(1/\sqrt{d})$. On the other hand, if we choose $n > d^{8t}$, the graph G^t will have girth at least $4t$, so the t-neighborhood of every vertex in G is a tree, and for every edge e in G^t, the union of t-neighborhoods of the endpoints of e in G is also a tree.

An assignment $\overline{\sigma} : V \to \Sigma^{d^t}$ in G^t describes, for each $v \in V$, v's "view" $\overline{\sigma}_v$ of assignments to vertices at distance at most t from v. Notice that for each $v \in V$ there are exactly two possibilities for $\overline{\sigma}_v$ that are consistent with local constraints. Namely, choose an arbitrary value (0 or 1) for v's view of itself, and propagate this assignment to v's view of its neighbors, their neighbors, etc., in a way that is consistent with the inequality constraints. For example, if $\overline{\sigma}_v(v) = 0$, then all vertices w at even distance from v (up to $2\lfloor t/2 \rfloor$) are assigned $\overline{\sigma}_v(w) = 0$, and all ws at odd distance from v are assigned $\overline{\sigma}_v(w) = 1$.

To show $\overline{\text{SAT}}(G^t) \geq 1/2$, we choose $\overline{\sigma}$ at random. That is, for each v, we choose between the two possibilities for $\overline{\sigma}_v$ by tossing a fair independent coin. Now for an arbitrary edge (u, v) of G^t, the assignments $\overline{\sigma}_u$ and $\overline{\sigma}_v$ will be consistent with probability 1/2, so this random $\overline{\sigma}$ satisfies half the constraints in expectation. It follows that there must exist an assignment satisfying half the constraints in G^t.

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References

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