Questions

1. How many $8 \times 8$ chessboard configurations are there with...

(a) 8 white rooks, and all must be in different rows and columns?

Solution: Such a configuration can be represented by a permutation of the numbers 1 to 8. Entry $i$ in the permutation represents the column of the rook in the $i$-th row. Therefore the number is $8!$.

(b) 4 white and 4 black rooks, and all must be in different rows and columns?

Solution: Such a configuration can be represented by a permutation and a sequence black-white colors with 4 blacks and 4 whites. By the product rule the number of configurations is $8! \cdot \binom{8}{4}$.

(c) 8 white and 8 black rooks, and there is exactly one white and one black in every row and every column? (Hint: Derangements.)

Solution: Once the positions of the white rooks are fixed, the placement of the black rooks can be described by a derangement of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The $i$-th entry of the derangement represents the column of the black rook in the row that has a white rook in column $i$ already. By the generalized product rule the number of configuration equals the number of permutations times the number of derangeements, namely $8!^2 (1 - 1/1! + 1/2! - \cdots + 1/8!)$.

2. This question concerns 5-card poker hands. Assume that all hands are equally likely. What is the probability that

(a) there are (at least) two cards with the same face value?

Solution: We consider the complement event, namely the event that no two cards have the same face value. There are 52 choices for the first card, 48 choices for the second card, 44 for the third card and so on. By the generalized product rule the number of ordered hands of this type is $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$. By the division rule the number of actual (unordered) hands is $52 \cdot 48 \cdot 44 \cdot 40 \cdot 36 / 5!$. Therefore the desired probability is

$$1 - \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36 / 5!}{\binom{52}{5}} = 1 - \frac{52 \cdot 48 \cdot 44 \cdot 40 \cdot 36}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}.$$

(b) exactly three of the four suits are represented?

Solution: We count separately the 3+1+1 hands that have 3 cards of one suit and one card of the other two suits, and 2+2+1 hands that have two cards of two suits and one card of a remaining suit. For the first type, there are 4 choices of the leading suit and 3 choices for the other two suits, and $\binom{13}{3} \cdot 13^2$ choices for the face values, so by the generalized product rule the number of hands is $12 \cdot \binom{13}{3} \cdot 13^2$. By a similar argument there are $12 \cdot \binom{13}{2}^2 \cdot 13$ hands of the second type. The total number is therefore

$$12 \cdot \binom{13}{3} \cdot 13^2 + 12 \cdot \binom{13}{2}^2 \cdot 13 \binom{52}{5}.$$
(c) There is a single card with the largest face value?

**Solution:** We count the number of hands in which there is a single largest face value and this value equals \( \ell \), where we think of \( \ell \) as a number between 1 and 13. There are 4 choices for the suit of the card with face value \( \ell \) and \( \binom{4(\ell-1)}{4} \) choices for the other cards’ face values, so the number is \( 4 \cdot \binom{4(\ell-1)}{4} \). Letting \( \ell \) range over all possible values we obtain that the desired probability equals \( 4 \cdot \left( \frac{\binom{3}{4} + \binom{8}{4} + \cdots + \binom{48}{4}}{\binom{52}{5}} \right) \).

(d) Your opponent has been dealt \{7♣,7♦,7♥,7♠,5♠\}. Your hand is chosen from the remaining cards. What is the probability you beat her? You need a four-of-a-kind with a face value larger than 7 or a straight flush (five consecutive cards of the same suit).

**Solution:** The sample space of interest consists of all \( \binom{47}{5} \) hands from the set of 47 remaining cards, and we assume equally likely outcomes. There are \( 7 \cdot 43 \) possible four-of-a-kinds with face value 8 or above (7 choices for the four-of-a-kind face value and 43 choices for the remaining card), \( 4 \cdot 3 \) straight flushes that do not involve a 5 (4 choices for the suit and 3 choices for the sequence of face values) and \( 3 \cdot 2 \) more that involve a 5 (3 choices for the suit and 2 for the sequence of face values). Your probability of winning is therefore \( 7 \cdot 43 + 4 \cdot 3 + 3 \cdot 2 \frac{\binom{47}{5}}{\binom{52}{5}} \).

3. In how many ways can you place 10 balls into 3 boxes if

(a) The balls and the boxes are labeled (i.e., distinguishable)?

**Solution:** There are 3 choices of box for each ball, so by the product rule the answer is \( 3^{10} \).

(b) The balls are indistinguishable and the boxes are labeled?

**Solution:** Such a configuration can be represented as a sequence of 10 stars and 2 bars, so there are \( \binom{12}{2} \) possible ways.

(c) The balls and the boxes are both indistinguishable?

(Hint: Show a bijection to the set \{\((x, y, z): 3x + 2y + z = 10, x, y, z \geq 0\)\}.)

**Solution:** A sequence of numbers \((x, y, z)\) such that \(3x + 2y + z = 10\) uniquely represents a configuration by assigning \(z, y + z\), and \(x + y + z\) balls to the three bins respectively. To count the number of such sequences we apply the sum rule. When \(x = 0\) there are six possible choices for \(y\) (0, 1, 2, 3, 4, 5) and \(z\) is determined. When \(x = 1\) there are four choices for \(y\) (0, 2, 4, 6), when \(x = 2\) there are three (0, 2, 4), and when \(x = 3\) there is only one (0), so there is a total of \(6 + 4 + 3 + 1 = 14\) configurations.

(d) The balls are labeled but the boxes are indistinguishable?

**Solution:** Let us first discount the configuration in which all balls fall in the same box. Let \(S\) be the set of configurations in which the boxes are distinguishable and at least two boxes are nonempty. Each configuration with indistinguishable boxes then arises from \(3! = 6\) different configurations in \(S\), so by the division rule this number equals \(|S|/6\). By part (a) and the sum rule the size of \(S\) equals \(3^{10} - 3\). Therefore the desired number is \((3^{10} - 3)/6 + 1\), where the plus one accounts for the discounted configuration. (Apologies the hint was misleading.)
This question concerns sequences of length $k$ whose entries are numbers between 1 and $n$.

(a) How many such sequences are there?

**Solution:** By the product rule there are $n^k$ such sequences.

(b) How many such sequences are there in which at least two entries are the same?

**Solution:** By the generalized product rule the number of such sequences in which all numbers are distinct is $n(n-1) \cdots (n-k+1)$, as there are $n$ first choices, $n-1$ second choices and so on. Therefore the desired number is $n^k - n(n-1) \cdots (n-k+1)$.

(c) If $n = 2$, how many sequences are there in which both a 1 and a 2 occur at least once?

**Solution:** There are exactly two sequences in which a 1 does not occur or a two does not occur. By the complement rule the desired number is $2^n - 2$.

(d) (Extra credit) If $k = 20$ and $n = 5$, how many sequences are there in which every number occurs at least once?

**Solution:** Let $N_i$ be the number of sequences in which number $i$ never occurs. Then the intersection of any of the sets $N_i$ is the set of sequences that exclude certain entries, so its size can be calculated by the product rule. Specifically, any $t$-wise intersection has size $(5-t)^{20}$. By inclusion-exclusion, the union $N_1 \cup \cdots \cup N_5$ has size $5 \cdot 4^{20} - \binom{5}{2} \cdot 3^{20} + \binom{5}{3} \cdot 2^{20} - \binom{5}{4} \cdot 1^{20}$. The number of interest is the size of the complement, namely

$$5^{20} - 5 \cdot 4^{20} + \binom{5}{2} \cdot 3^{20} - \binom{5}{3} \cdot 2^{20} + \binom{5}{4}.$$

This is true because the left-hand side equals $|A_1 \cap A_{n+1}| + \cdots + |A_n \cap A_{n+1}|$, and a union of sets can never be larger than the sum of their sizes.

(b) Suppose $A_1, \ldots, A_n$ are all of the same size $s$, they are all subsets of $\mathcal{X}$, and each pair intersects in at most one element. Show that $|\mathcal{X}| \geq n \cdot s - \binom{n}{2}$.

**Solution:** On the other hand $A_1 \cup A_n$ is a subset of $\mathcal{X}$. On the other hand $|A_i| = s$ for every $i$ and $|A_i \cap A_j| = 1$ for every $i \neq j$. Plugging into (a) we obtain that $|\mathcal{X}| \geq n \cdot s - \binom{n}{2}$.

(c) Let $\mathcal{X}$ be the set of all subsets of size 2 of $\{1, \ldots, t\}$ and $A_i = \{S \in \mathcal{X} : i \in S\}$. Show that the inequality in part (b) is an equality for these sets.

**Solution:** $\mathcal{X}$ has size $\binom{t}{2}$, and there are $t$ sets $A_1, \ldots, A_t$, each of size $t-1$ (as there are exactly this many choices for the other entry of $S$) and $A_i \cap A_j = \{i, j\}$, so their intersection size is one. Since $\binom{t}{2}$ equals $t(t-1) - \binom{t}{2}$ the inequality in part (b) is tight.
6. In this question you will investigate the best possible advantage for the second player in the game of intransitive dice. Given a set $D$ of three dice, Alice chooses a die, Bob chooses one of the remaining two dice, then they each toss their die and the higher number wins. Let $p(D)$ be the probability that Bob wins, assuming they both choose their die optimally.

(a) Calculate the value of $p(D)$ for the following three dice:

Die A: 3, 3, 3, 3, 3, 6  
Die B: 2, 2, 2, 5, 5, 5  
Die C: 1, 4, 4, 4, 4, 4.

Solution: Out of the 36 possible outcomes for each pair of dice, there are 21 in which die $A$ beats die $B$, 21 in which die $B$ beats die $C$, and 25 in which die $C$ beats die $A$. So Bob can always choose a die that beats Alice’s die with probability at least $21/36 = 7/12$.

(b) Let $D = \{A, B, C\}$ be any set of three $n$-sided dice. Consider the experiment in which all three dice are tossed. Describe the sample space and the probabilities of all possible outcomes.

Solution: The sample space consists of all triples $(i, j, k)$, where $1 \leq i, j, k \leq n$, representing the face of die $A$, $B$, and $C$, respectively. All $n^3$ outcomes are equally likely, so each has probability $1/n^3$. The event that die $A$ beats die $B$ consists of those triples in which the value on face $i$ is larger than the value on face $j$, and so on.

(c) If the face values of the outcome are $(a, b, c)$, let $E_{AB}$ be the event $a > b$, $E_{ABC}$ be the event $a > b > c$, and so on. Assume the dice have disjoint sets of face values. Show that

$$\Pr[E_{AB}] + \Pr[E_{BC}] = 1 + \Pr[E_{ABC}] - \Pr[E_{CBA}].$$

Solution: By the inclusion-exclusion principle for probabilities,

$$\Pr[E_{AB} \cup E_{BC}] = \Pr[E_{AB}] + \Pr[E_{BC}] - \Pr[E_{AB} \cap E_{BC}].$$

The event $E_{AB} \cap E_{BC}$ consists of those outcomes in which die $A$ beats die $B$ and die $B$ beats die $C$, namely the event $E_{ABC}$. The event $E_{AB} \cup E_{BC}$ consists of those outcomes in which die $A$ beats die $B$ or die $B$ beats die $C$. Its complement is the event in which die $B$ beats die $A$ and die $C$ beats die $B$, namely the event $E_{CBA}$. Therefore $\Pr[E_{AB} \cup E_{BC}] = 1 - \Pr[E_{CBA}]$. Therefore

$$1 - \Pr[E_{CBA}] = \Pr[E_{AB}] + \Pr[E_{BC}] - \Pr[E_{ABC}].$$

The desired identity is obtained by rearranging terms.

(d) Use part (c) to show that for every $D$, $p(D) \leq 2/3$.

Solution: We prove this by contradiction. Assume that there exist a $D$ such that $p(D) > 2/3$. We consider two cases.

If $\Pr[E_{AB}] > 2/3$, then $\Pr[E_{CA}]$ must also be greater than $2/3$; if not, then Alice could win with probability at least $1/3$ by playing die $A$. By the same reasoning $\Pr[E_{BC}]$ must also be greater than $2/3$. By the inequality in part (b), it follows that all of $\Pr[E_{ABC}]$, $\Pr[E_{CBA}]$, $\Pr[E_{CAB}]$ are greater than $1/3$. Since the events $E_{ABC}$, $E_{BCA}$, and $E_{CAB}$ are disjoint, it follows that $\Pr[E_{ABC} \cup E_{BCA} \cup E_{CAB}] > 1$, a contradiction.

If $\Pr[E_{AB}] < 2/3$, then $\Pr[E_{BA}] \geq 1/3$, so $\Pr[E_{CB}]$ must be greater than $2/3$; if not Alice could win with probability at least $2/3$ by playing die $B$. By the same reasoning as in case one we can conclude that $\Pr[E_{CB}]$, $\Pr[E_{BA}]$, $\Pr[E_{AC}]$ are all greater than $2/3$ and obtain a contradiction by the same argument.

(e) (Extra credit) Can you find a set of dice (no restriction on the number of faces) for which $p(D)$ is larger than the value you calculated in part (a)?

Solution: It is possible, but we’ll leave this for some other time.