1. Consider 3 boys (Xavier, Yoshi, and Zorba) and 3 girls (Aerith, Bowie, and Chrissie) with the following preference lists:

<table>
<thead>
<tr>
<th></th>
<th>Xavier</th>
<th>Yoshi</th>
<th>Zorba</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>2 3 1</td>
<td>3 1 2</td>
<td>1 2 3</td>
</tr>
<tr>
<td>2nd</td>
<td>2 1 3</td>
<td>1 3 2</td>
<td>3 2 1</td>
</tr>
<tr>
<td>3rd</td>
<td>Aerith</td>
<td>Bowie</td>
<td>Chrissie</td>
</tr>
</tbody>
</table>

(a) Describe the execution of Gale-Shapley algorithm on these preference lists with the boys proposing. Show the stable matching produced by the algorithm.

(b) Repeat part (a), but now with the girls proposing.

(c) Is there a stable matching different both from the one in part (a) and the one in part (b)? Justify your answer.

Solution:

(a) In the first round, Xavier, Yoshi and Zorba propose to Chrissie, Bowie and Zorba respectively. As every girl has only one proposal, she accepts the proposal and the algorithm ends in one round. Here is the output when boys are proposing:

The stable matching is: Xavier to Chrissie, Yoshi to Bowie, Zorba to Aerith.

(b) In the first round, Aerith, Bowie and Chrissie propose to Yoshi, Xavier and Zorba respectively. As every boy has only one proposal, he accepts the proposal and the algorithm ends in one round. Here is the output when girls are proposing:

The stable matching is: Aerith to Yoshi, Bowie to Xavier, Chrissie to Zorba.

(c) Yes, consider the matching:
We can see that the 6 potential rogue pairs (Xavier, Bowie), (Xavier, Chrissie), (Yoshi, Aerith), (Yoshi, Bowie), (Zorba, Aerith), (Zorba, Chrissie) all consist of preference score 3, while the above matching has preference score 2 for everyone. So no one would have incentive to switch. Therefore this is a stable matching, different from the ones in part (a) and part (b).

2. Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.

(a) \(1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n\)
(b) \(1^3 + 2^3 + \cdots + n^3\)
(c) \(n \cdot 1^2 + (n - 1) \cdot 2^2 + \cdots + 1 \cdot n^2\)
(d) (Extra credit) \((0^2 + \cdots + n^2) + (1^2 + \cdots + (n + 1)^2) + \cdots + (n^2 + \cdots + (2n)^2)\)

Solution:

(a) Let \(S(n) = 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n\). Then
\[
2S(n) = 1 \cdot 2^2 + 2 \cdot 2^3 + \cdots + (n \cdot 2^n + n \cdot 2^{n+1})
\]
\[
S(n) = 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n
\]
Subtracting the second formula from the first we obtain that
\[
S(n) = 2S(n) - S(n) = n \cdot 2^{n+1} - 1 - 2^1 - 2^2 - \cdots - 2^n + 1
\]
Using the formula for a geometric sum, we can simplify the expression to
\[
S(n) = n \cdot 2^{n+1} - \frac{2^{n+1} - 1}{2 - 1} + 1 = (n - 1) \cdot 2^{n+1} + 2
\]

(b) Let \(S(n) = 0^3 + 1^3 + 2^3 + \cdots + n^3\). This is an expression that involves cubes, so we guess a solution of the form \(S(n) = an^4 + bn^3 + cn^2 + dn + e\) for some numbers \(a, b, c, d, e\). If our guess is correct, the evaluation \(S(0) = 0\) gives that \(e = 0\), and the evaluations \(S(1) = 1, S(2) = 9, S(3) = 36, S(4) = 100\) give rise to the following system of linear equations:
\[
a + b + c + d = 1
16a + 8b + 4c + 2d = 9
81a + 27b + 9c + 3d = 36
256a + 64b + 16c + 4d = 100
\]
This system has the unique solution \(a = 1/4, b = 1/2, c = 1/4, d = 0\).
We now prove that, in fact, $S(n) = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{1}{4}n^2 \cdot (n^2 + 2n + 1) = \frac{1}{4}n^2(n + 1)^2$ by induction on $n$. When $n = 0$, $S(n) = 0$ and $\frac{1}{4}n^2(n + 1)^2 = 0$ as desired. Now we assume $S(n) = \frac{1}{4}n^2(n + 1)^2$ for some $n$. Then

$$S(n + 1) = S(n) + (n + 1)^3 = \frac{1}{4}n^2(n + 1)^2 + (n + 1)^3 = \frac{1}{4}(n^2 + 4n + 4)(n + 1)^2$$

$$= \frac{1}{4}(n + 2)^2(n + 1)^2$$

as desired.

(c) Let $S(n) = n \cdot 1^2 + \cdots + 1 \cdot n^2 = (1^2) + (1^2 + 2^2) + \cdots + (1^2 + \cdots + n^2)$. By the sum-of-squares formula from class and the result from part (b),

$$S(n) = \left(\frac{1}{3} \cdot 1^3 + \frac{1}{2} \cdot 1^2 + \frac{1}{6} \cdot 1\right) + \cdots + \left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right)$$

$$= \frac{1}{3}(1^3 + \cdots + n^3) + \frac{1}{2}(1^2 + \cdots + n^2) + \frac{1}{6}(1 + \cdots + n)$$

$$= \frac{1}{3}(\frac{1}{4}n^2(n + 1)^2) + \frac{1}{2}(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n) + \frac{1}{6}(\frac{1}{2}n(n + 1))$$

$$= n^4(\frac{1}{12}) + n^3(\frac{1}{6} + \frac{1}{6}) + n^2(\frac{1}{12} + \frac{1}{4} + \frac{1}{12}) + n(\frac{1}{12} + \frac{1}{12})$$

$$= \frac{1}{12}n^4 + \frac{1}{3}n^3 + \frac{5}{12}n^2 + \frac{1}{6}n$$

(d) Let $S(n) = (0^2 + \cdots + n^2) + \cdots + (n^2 + \cdots + (2n)^2)$, and $S_c(n) = n \cdot 1^2 + \cdots + 1 \cdot n^2$ be the sum in (c). Then,

$$S(n) = (1^2 + \cdots + n^2) + (1^2 + \cdots + (n + 1)^2) + \cdots + (1^2 + \cdots + (2n)^2)$$

$$- (n - 1) \cdot 1^2 - (n - 2) \cdot 2^2 - \cdots - 1 \cdot (n - 1)^2$$

$$= (0^2 + \cdots + n^2) + \cdots + (0^2 + \cdots + (2n)^2) - S_c(n - 1)$$

$$= S_c(2n) - S_c(n - 1) - S_c(n - 1)$$

$$= \frac{1}{3}n^4 + \frac{5}{9}n^3 + \frac{5}{6}n^2 + \frac{1}{3}n - 2\left(\frac{1}{12}n(n - 1)^4 + \frac{1}{3}(n - 1)^3 + \frac{5}{12}(n - 1)^2 + \frac{1}{6}(n - 1)\right)$$

$$= \frac{4}{3}n^4 + \frac{8}{9}n^3 + \frac{5}{6}n^2 + \frac{1}{3}n - 2\left(\frac{1}{6}n^4 + \frac{1}{6}n^2\right)$$

$$= \frac{7}{6}n^4 + \frac{1}{9}n^3 + \frac{1}{3}n^2 + \frac{1}{3}n$$

3. Show the following inequalities.

(a) 1.222 \leq 1 + 3^{-2} + 5^{-2} + 7^{-2} + \cdots \leq 1.252
(b) \sqrt{2} - 2 + \frac{1}{\sqrt{2}} \leq 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1
(c) 1 - 2(n + 1)^{-1} \leq \frac{4}{n(n+1)(n+2)} \leq 1 - 2(n + 2)^{-2}
   (Hint: \frac{1}{x(x+2)} - \frac{1}{x(x+1)} = \frac{1}{x(x+1)} - \frac{1}{x(x+2)})

Solution:

(a) We split the summation like this:

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2}\right) + \left(\frac{1}{11^2} + \frac{1}{13^2} + \cdots \right).$$

The first part is between 1.183 and 1.184. We bound the second part by the corresponding integral:

\[
\begin{array}{c}
\int_{11}^{13} \frac{1}{x^2} dx \\
\int_{13}^{15} \frac{1}{x^2} dx \\
\int_{15}^{17} \frac{1}{x^2} dx
\end{array}
\]

\[x\]
The area under the boxes equals $\frac{2}{11^2} + \frac{2}{13^2} + \cdots$, which is at least as large as the integral $\int_{11}^{\infty} \frac{1}{x^2} \, dx = \frac{1}{11}$. On the other hand, it is no larger than the integral minus the lightly shaded area, which is $\frac{1}{11} - \frac{1}{11^2}$. Therefore

$$\frac{1}{2 \cdot 11} - \frac{1}{2 \cdot 11^2} \leq \frac{1}{9^2} + \frac{1}{11^2} + \cdots \leq \frac{1}{2 \cdot 11},$$

and so

$$1.183 + \frac{1}{22} - \frac{1}{242} \leq 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \leq 1.184 + \frac{1}{22}.$$  

The number on the left is larger than 1.223, while the one on the right is smaller than 1.228, so the sum is within the desired bounds.

(b) The value of the sum from 1 to $n-1$ equals the area under the first $n-1$ bars.

\[ \frac{1}{\sqrt{x}} \]

This area is at least the integral of the function $\frac{1}{\sqrt{x}} = x^{-1/2}$ from 1 to $n$:

$$1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}} \geq \int_{1}^{n} x^{-1/2} \, dx = 2x^{1/2} \bigg|_{1}^{n} = 2\sqrt{n} - 2.$$  

Subtracting the area of the light boxes, which equals $1 - 1/\sqrt{n}$, we get that the sum on the left is at most $2\sqrt{n} - 1 - 1/\sqrt{n}$, that is

$$2\sqrt{n} - 1 - \frac{1}{\sqrt{n}} \leq 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n-1}} \leq 2\sqrt{n} - 2.$$  

If we add $1/\sqrt{n}$ to all terms we obtain the desired inequality.

(c) Since $\frac{4}{x(x+1)(x+2)}$ equals $\frac{2}{x(x+1)} + \frac{2}{(x+1)(x+2)}$, the sum of interest simplifies to

$$\frac{2}{1 \cdot 2} - \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3} - \frac{2}{3 \cdot 4} + \cdots + \frac{2}{n \cdot (n+1)} - \frac{2}{(n+1) \cdot (n+2)}.$$  

All the terms cancel out except the first one and the last one, so the sum equals $1 - \frac{2}{2/(n+1)(n+2)}$, which is always within the desired bounds.

4. Are the following propositions about digraphs true or false? Justify your answer by giving a proof or a relevant example.

(a) The sum of in-degrees of a digraph equals the sum of out-degrees. The in-degree and out-degree of a vertex is its number of incoming and outgoing edges, respectively.

(b) There exists a balanced DAG with at least one edge. A digraph is balanced if for every vertex $v$ the in-degree of $v$ equals the out-degree of $v$.

(c) For every $n \geq 1$, there is a digraph with vertices 1, \ldots, $n$ such that the in-degree of vertex $v$ is $n - v$ and its out-degree is $v - 1$.

Solution:
(a) True. Both are equal to the number of edges. The edges can be counted by going over all
the vertices \( v \) and counting the edges that point to \( v \), which is the sum of the in-degrees.
They can also be counted by going over all the vertices \( w \) and counting the edges that
point away from \( w \), which is the sum of the out-degrees.
(b) False. We disregard the isolated vertices (if any). The DAG on the remaining vertices
must have a sink, which has positive in-degree and out-degree zero.
(c) True. Take the digraph of \( n \) vertices where \((v, w)\) is an edge if and only if
\( v > w \). This is a DAG because the sequence \( n, (n−1), \ldots, 1 \) is a topological sort. Vertex \( v \) has incoming
edges from \( 1, \ldots, v−1 \) and outgoing edges to \( v+1, \ldots, n \), so the degree requirements
are met.

5. A \( k \)-regular forest is a forest all of whose vertices have degree 1 or degree \( k \).

(a) Prove that if \( G \) is a 3-regular forest then \( n = 2(\ell−c) \) where \( n, c, \ell \) is the number of
vertices, connected components, and leaves in \( G \), respectively. (Hint: Induction on \( n \).)
(b) Prove that if \( G \) is a \( k \)-regular forest, then \( (k−2)n = (k−1)\ell−2c \), and \( n \) or \( \ell \) must be
even, for every \( k \geq 3 \).
(c) Prove that a \( k \)-regular tree (i.e., \( c = 1 \)) \( G \) exists if and only if \( n−2 \) is a multiple of \( k−1 \).
(Hint: Apply induction on the number of degree-\( k \) vertices)

Solution:

(a) First assume \( G \) is a 3-regular tree. On the one hand, \( G \) has \( n−1 \) edges. On the other
hand, the number of edges is half the sum of the degrees, which equals \( 3(n−\ell)+1 \cdot \ell \).
Therefore \( 3(n−\ell)+\ell = 2(n−1) \), so \( n = 2\ell−2 \). If \( G \) is a forest, this equation holds for every
connected component of \( G \). Summing up all of them we get that \( n = 2\ell−2c = 2(\ell−c) \).
(b) By the same reasoning as in part (a), if \( G \) is a \( k \)-regular tree then \( k(n−\ell)+\ell = 2(n−1) \),
from where \( (k−2)n = (k−1)\ell−2 \). Summing up over all components we get the desired
equation.
(c) If \( G \) exists, by part (b) we have \( (k−2)n = (k−1)\ell−2 \), so \( n−2 = (n−\ell)(k−1) \). Now
we prove that if \( n−2 \) is a multiple of \( k−1 \) then \( G \) exists. The proof is by induction on
the number \( d = n−\ell \) of internal vertices. The base case is \( d = 0 \), in which case \( G \) is
a tree of two vertices with one edge between them and \( n = 2 \) as desired. Now assume
there is a \( k \)-regular tree \( G \) with \( n = d(k−1) + 2 \) vertices. Take an arbitrary leaf of \( G \)
and attach \( k \) new leaves to it. The new graph has \( d+1 \) internal vertices and \( n+(k−1) \)
total vertices, so its number of vertices equals \( d(k−1) + 2 + (k−1) = (d+1)(k−1) + 2 \)
as desired.

6. Let \( G \) be the digraph whose vertices are the 125 3-digit numbers with digits 0, 1, 2, 3, 4
(leading zeros are allowed), and \((u, v)\) is an edge if \( v−u \) equals 1, 10, or 100.

(a) Show that \( G \) is acyclic.
(b) What is the duration of the shortest parallel schedule for \( G \)? Justify your answer.
(c) Show that \( G \) has an antichain of size 19.
(d) Show that the vertices of \( G \) can be partitioned into 19 (vertex-disjoint) paths. Conclude
that \( G \) does not have an antichain of size 20.

Solution:

(a) Order the numbers from smallest to largest. This ordering is a topological sort: There
cannot be no back edges because the differences in the backward direction are all negative.
(b) \( G \) has a path of length 12, namely the path that starts at 000 and increases the digits one by one (in any order). On the other hand, \( G \) cannot have a path of length 13 because along every edge at least one of the digits must increase, and this can happen at most four times per digit. By Theorem 10 in Lecture 6, \( G \) has a parallel schedule of duration at most 12, and this is the best possible.

(c) If there is a path from \( v \) to \( w \) then the sum of digits of \( w \) must be greater than the sum of digits of \( v \). Therefore the set of vertices whose digits sum up to 6 is an antichain. There are 19 such vertices: 3 that start with 0/4, 4 that start with 1/3, and 5 that start with 2.

(d) The following 19 paths are vertex-disjoint and partition all 125 vertices:

\[
\begin{align*}
p_a &= (a00, a01, a02, a03, a04, a14, a24, a34, a44), & \text{where } a &\in \{0, 1, 2, 3, 4\} \\
q_a &= (a10, a11, a12, a13, a23, a33, a43), & \text{where } a &\in \{0, 1, 2, 3, 4\} \\
r_{bc} &= (0bc, 1bc, 2bc, 3bc, 4bc), & \text{where } b &\in \{2, 3, 4\} \text{ and } c &\in \{0, 1, 2\}
\end{align*}
\]

If \( G \) had an antichain of size 20, then two of its vertices would have to belong to one of these 19 paths. However, no two vertices in an antichain can belong to the same path.

**ESTR 2004 mini-project**  
Consider the following types of graphs on \( n \) vertices:

- A type A graph is the union of two edge-disjoint cycles, each of length \( n \). (The two cycles cannot share any edges.)
- A type B graph is the union of two distinct cycles, each of length \( n \). (The two cycles may share some edges but they are not identical.)

Let \( s_A(n) \) and \( s_B(n) \) be the maximum possible size of a shortest cycle among all type A graphs and type B graphs on \( n \) vertices, respectively.

(a) What is \( s_A(8) \) and \( s_B(8) \)?
(b) Calculate the exact values of \( s_A(n) \) and \( s_B(n) \) for as many values of \( n \) as you can.
(c) Obtain the best upper and lower bounds you can for \( s_A(9, 900) \) and \( s_B(9, 900) \).

**Solution:**

(a) We will show that \( s_A(8) = s_B(8) = 4 \). As there are more type B graphs than type A graphs on \( n \) vertices, \( s_B(n) \) is at least as large as \( s_A(n) \), so it suffices to show that \( s_A(4) \geq 4 \) and \( s_B(4) \leq 4 \).

The following is a diagram of a type A graph on 8 vertices. As the graph is bipartite, it cannot have cycles of length less than 4.

![Diagram of a type A graph on 8 vertices]

We now argue that a type B graph on 8 vertices must have a cycle of length at most 4. We may label the vertices of the graph by the residues modulo 8 so that the edges of the first cycle are those pairs \( \{x, y\} \) such that \( x - y \equiv \pm 1 \pmod{8} \). Let \( D \) be the set of edges of the second cycle that are not present in the first cycle. It is impossible to have \( |D| = 1 \) for then the second cycle would not cover all the vertices. If \( |D| = 2 \) then the two edges
in $D$ must have the form $\{x, y\}$ and $\{x - 1, y + 1\}$ modulo 8, in which case the sequence $x - 1, x, y, y + 1$ is a cycle of length 4. If $|D| \geq 3$, assume for contradiction that $D$ has no cycle of length less than 4. Then all edges in $D$ must be of the form $\{x, x + 4\}$ modulo 8. Therefore all edges in $D$ intersect $\{0, 1, 2, 3\}$ at distinct vertices. Since there are at least three such edges, two of these intersection vertices must form an edge $\{x, x + 1\}$. Then $\{x, x + 1, x + 4, x + 5\}$ is a cycle of length 4 in $G$, contradicting our assumption.

(c) We will show the following bounds for every $n$:

$$s_A(n) \leq \lfloor 2 \log_3((n + 1)/2) \rfloor + 1 \quad s_B(n) = \lfloor n/3 \rfloor + 2 \text{ for } n \geq 6$$

In particular, $s_A(9, 900) \leq 16$ and $s_B(9, 900) = 3, 302.$

To show the first one, let us assume that a type A graph on $n$ vertices does not have a cycle of length $2\ell$ or less, where $\ell$ is an integer. Take any vertex $r$ in this graph. Then the subgraph induced by taking all paths originating from $r$ of length at most $\ell$ must be a tree. All internal vertices in this tree have degree 4. Therefore the tree must have

$$1 + 4 + 4 \cdot 3 + 4 \cdot 3^2 + \cdots + 4 \cdot 3^{\ell - 1} = 2 \cdot 3^\ell - 1$$

vertices. Each of these vertices represents a distinct vertex in the graph, so it must be that $2 \cdot 3^\ell - 1 \leq n$, or $\ell \leq \log_3((n + 1)/2)$.

To summarize, we argued that if a type A graph on $n$ vertices does not contain a cycle of length $2\ell$ or less, then $2\ell \leq 2 \log_3((n + 1)/2)$. In the contrapositive, if $2\ell > 2 \log_3((n + 1)/2)$ then the graph must have a cycle of length at most $2\ell$.

We now show that $s_B(n) \geq \lfloor n/3 \rfloor + 2$. The following graph on $n$ vertices contains no shorter cycle: The vertices are the integers from 0 to $n - 1$ modulo $n$. The first cycle visits the vertices in increasing order. To describe the second cycle let $a = 0, b = \lfloor n/3 \rfloor$ and $c = n - \lfloor n/3 \rfloor$. The second cycle is

$$a + 1 \sim b, c + 1 \sim a, b + 1 \sim c$$

where $x \sim y$ is a shorthand for the vertex sequence $x, x + 1, \ldots, y$ modulo $n$. Any cycle in the graph must then contain at least one of the sequences $a + 1 \sim b, b + 1 \sim c$, or $c + 1 \sim a$ (which have length at least $\lfloor n/3 \rfloor - 1$ each) plus at least three extra edges.

Finally, we show that $s_B(n) \leq \lfloor n/3 \rfloor + 2$ for all $n \geq 6$. As in part (a), label the vertices by the residues modulo $n$ and represent the edges of the first cycle by the pairs $\{x, x + 1\}$ modulo $n$. Let $D$ be the set of edges in the second cycle that are not present in the first. If $|D| = 2$ then the graph has a cycle of length 4 which is at most $\lfloor n/3 \rfloor + 2$, so let us assume that $|D| \geq 3$. We will call two edges $\{x, y\}$ and $\{x', y'\}$ intersecting if $x \leq x' \leq y \leq y'$ under the standard ordering of the integers from 0 to $n - 1$. We now consider two cases.

If $D$ contains two non-intersecting edges $\{x_1, y_1\}, \{x_2, y_2\}$ then we may assume, up to relabeling, that (1) $x_1 \leq x_2 \leq y_2 \leq y_1$ or (2) $x_1 \leq y_1 \leq x_2 \leq y_2$. In case (1), the three cycles

$$x_2 \sim y_2 \quad y_1 \sim x_1 \quad x_1 \sim x_2, y_2 \sim y_1$$

altogether cover each edge in the first cycle once and the other two edges twice each. This is a total of $n + 4$ edges with multiplicities, so one of the three cycles has no more than $\lfloor (n + 4)/3 \rfloor \leq \lfloor n/3 \rfloor + 2$ edges. In case (2), the three cycles

$$x_1 \sim y_1 \quad x_2 \sim y_2 \quad y_1 \sim x_2, y_2 \sim x_1$$

have the same property, so one of them must again have no more than $\lfloor n/3 \rfloor + 2$ edges.
If, on the other hand, every pair of edges in $D$ intersect then $D$ must contain three edges \{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\} such that $x_1 \leq x_2 \leq x_3 \leq y_1 \leq y_2 \leq y_3$. In this case, the cycles

\[
x_1 \rightsquigarrow x_2 \rightsquigarrow y_1 \quad x_2 \rightsquigarrow x_3 \rightsquigarrow y_2 \quad x_3 \rightsquigarrow x_1 \rightsquigarrow y_3
\]

altogether cover each edge in the first cycle once and the other three edges twice each. This is a total of $n + 6$ edges with multiplicities, so one of the three cycles has no more than $\lfloor (n + 6)/3 \rfloor = \lfloor n/3 \rfloor + 2$ edges as desired.
ESTR 2004 mini-project: Solution