1. There are $n$ boys and $n$ girls. In how many ways can you

(a) divide them into $n$ \{boy, girl\} pairs?
(b) divide them into $n$ unordered pairs without constraints on gender?
(c) divide them into $n$ unordered pairs of the same gender? Assume $n$ is even.

Solution:

(a) Each permutation of the girls uniquely specifies such a pairing: The first boy is matched to the first girl in the permutation, the second boy is matched to the second girl in the permutation, and so on. So the number of such pairings equals the number of permutations of the girls which is $n!$.

(b) As genders are irrelevant we need to count the number of ways to divide $2n$ people into $n$ pairs. If we order the pairs, the first pair can be chosen in $\binom{2n}{2}$ ways. Once it is chosen the second pair can be chosen in $\binom{2n-2}{2}$ ways and so on, so by the generalized product rule, the total number of ways to divide the people into $n$ pairs that are ordered is

$$\binom{2n}{2} \cdot \binom{2n-2}{2} \cdots \binom{2}{2} = \frac{(2n)(2n-1)}{2} \cdot \frac{(2n-2)(2n-3)}{2} \cdots \frac{2 \cdot 1}{2} = \frac{(2n)!}{2^n}.$$ 

As there are $n!$ ways to order the pairs, by the division rule the desired count is $\frac{(2n)!}{(n!2^n)}$.

Here is another way to answer this question. Let $T(2n)$ be the number of ways to pair up $2n$ people. A partner for the first person can be chosen in $2n - 1$ ways. Once this is done $2n - 2$ people remain to be paired up. So by the generalized product rule, $T(2n) = (2n - 1) \cdot T(2n - 2)$. Since $T(2) = 1$, it follows that

$$T(2n) = (2n - 1) \cdot (2n - 3) \cdot (2n - 5) \cdots 1.$$ 

This is the same number as $\frac{(2n)!}{(n!2^n)}$. Can you see why?

(c) By part (b), the boys can be paired up in $\frac{n!}{((n/2)!)^2}$ ways and so can the girls. By the product rule the number of possible pairings is

$$\left(\frac{n!}{(n/2)! \cdot 2^{n/2}}\right)^2 = \frac{n!^2}{(n/2)!^2 2^{n/2}}.$$ 

2. This question concerns 5-card poker hands. Assume that all hands are equally likely. What is the probability that the hand has

(a) an odd number of hearts?
(b) exactly two cards with the same face value?
(c) at least one heart and at least one spade, and the face value of the largest heart is the largest value of the largest spade?
(d) Your opponent has been dealt \{9♠, 9♦, 9♥, 9♣, 5♠\}. Your hand is chosen from the remaining cards. What is the probability you beat her? You need a four-of-a-kind with a face value larger than 9 or a straight flush (five consecutive cards of the same suit).
Solution:

(a) The sample space consists of all 5-card subsets of the 52-card deck. The size of the sample space is \( \binom{52}{5} = 2,598,960 \), so the probability of any event \( E \) is \( \Pr[E] = |E|/\binom{52}{5} \).

Let \( S \) denote the set of hands with an odd number of hearts and \( A_i \) denote the set of hands with exactly \( i \) hearts. By the sum rule, the cardinality of \( S \) is the sum of the cardinalities of \( A_1, A_2 \) and \( A_3 \). \( |A_1| = \binom{13}{1} \binom{39}{4} = 1,069,263. \ |A_3| = \binom{13}{3} \binom{26}{2} = 211,926. \ |A_5| = \binom{13}{5} = 1,287, \) so \( \Pr[S] = \frac{1,069,263 + 211,926 + 1,287}{2,598,960} \approx 0.49 \).

(b) Let \( S \) denote the set of hands with exactly two cards with the same face value. There are 13 ways to choose this face value, \( \binom{2}{2} \) ways to choose the suits of these two cards, \( \binom{12}{3} \) ways to choose the (distinct) face values of the other three cards, and for each of them, 4 ways to choose a suit. By the generalized product rule, \( |S| = \binom{13}{1} \binom{2}{2} \binom{12}{3} \binom{4}{1}^3 = 1,098,240 \) and \( \Pr[S] \approx 0.42 \).

(c) The question statement is imprecise so we consider two events \( A \) and \( B \). Either one of these counts as a correct answer. We will identify face values with the numbers 1 to 13.

Let \( A \) be the event that the largest heart and largest spade face values are equal and \( A_i \) be the event that both of these are equal to \( i \). Then \( A \) is a disjoint union of the sets \( A_1 \) up to \( A_{13} \), so

\[ |A| = |A_1| + |A_2| + \cdots + |A_{13}|. \]

The hands in \( A_i \) must contain the \( i \) of hearts, the \( i \) of spades, and three more cards chosen from the hearts and spades up to face value \( i - 1 \) plus any of the 26 other cards. Therefore \( |A_i| = \binom{2(1+i)-26}{3} \) and so

\[ |A| = \binom{26}{3} + \binom{28}{3} + \binom{30}{3} + \cdots + \binom{50}{3} = 123,136. \]

from where \( \Pr[A] \approx 0.047 \).

Let \( B \) be the event that the largest heart is strictly bigger than the largest spade, and there is at least one of each. Write \( C \) for the event that the largest spade is strictly bigger than the largest heart and there is at least one of each. By symmetry, \( |B| = |C| \). The disjoint union \( S = A \cup B \cup C \) is the event that there is at least one heart and at least one spade. By the sum rule,

\[ |S| = |A| + |B| + |C| = |A| + 2|B| \]

so \( |B| = (|S| - |A|)/2 \). The complement \( \overline{S} \) consists of those hands with no hearts or no spades. By inclusion-exclusion, \( |\overline{S}| = \binom{3-13}{5} + \binom{3-13}{5} - \binom{2-13}{5} = 1,085,734 \). Therefore \( |S| = 1,513,226 \). So \( |B| = (1,513,226 - 123,136)/2 = 695,045 \) and \( \Pr[B] \approx 0.27 \).

(d) The sample space now consists of all 5-card hands from the 47-card deck that excludes your opponent’s hand. The outcomes are equally likely so the probability of an event \( E \) is \( |E|/\binom{47}{5} \). You can beat your opponent with a four-of-a-kind with face value larger than 9, or with a straight flush. Let \( A \) and \( B \) be the respective events. For the hands in \( A \), the four-of-a-kind face value can be chosen in five ways and the remaining card in 33 ways, so by the generalized product rule \( |A| = 5 \cdot 33 = 165 \). For the hands in \( B \), there are eight possible straight flushes in clubs, diamonds, and hearts, and four in spades, so by the sum rule \( |B| = 3 \cdot 8 + 4 = 28 \). So \( |A \cup B| = 193 \) and \( \Pr[A \cup B] = 193/\binom{47}{5} \approx 0.00013 \).

3. What is the probability of the seating arrangements happen with four boys and six girls in the following table settings? Assume all arrangements are equally likely.

(a) They sit in a line and no two boys sit together.
(b) They sit in a circle and no two boys sit together. (Two arrangements are identical if they differ by a turn of the circle.)

c) They sit in a line and no three girls sit together.

Solution:

(a) The cardinality of the sample space is $10!$. There are $6!$ permutations of the girls. Four boys are inserted into the gaps between the girls and no two boys are inserted into the same gap. There are $\binom{7}{4}4!$ ways of insertion. Therefore the probability that they sit in a line and no two boys sit together is $6!\binom{7}{4}4!/10! = 1/6 \approx 0.167$.

(b) The cardinality of the sample space is $10!/10 = 9!$. The number of possible circular arrangements of the girls around the table is $5!$. Once this is fixed, the first boy can be placed in any of the 6 gaps between the girls, the second boy can be placed in any of the 5 remaining gaps, and so on, for a total of $5! \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 43,200$ possibilities. So the probability of this type of sitting arrangement is $43,200/9! = 5/42 \approx 0.119$.

c) Let $A_i$ be the set of arrangements in which no three girls sit together and there are exactly $i$ blocks of girls (each of which can be a single girl, or a pair of girls). Then $A$ is the disjoint union of $A_3$, $A_4$, and $A_5$. In the set $A_3$, the girls are separated into three pairs. In $A_4$ there are two pairs and the other two girls sit alone. In $A_5$ there is one pair and the other four girls sit alone.

To calculate the size of $A_3$, we first choose the positions of the girl-seat-blocks in $\binom{5}{3}$ possible ways among the five possible gaps separated by boy-seats. Once these are fixed, the arrangement is specified by the $4!$ possible assignments of boys to their seats and the $6!$ assignments of girls to their seats, so $|A_3| = \binom{5}{3} \cdot 4! \cdot 6!$. For $A_4$, the girl-seat-blocks can be chosen in $\binom{2}{2}$ ways, the identity of the pairs among these in $\binom{4}{2}$ ways, and the assignment of people to seats in $4! \cdot 6!$ ways, so $|A_4| = \binom{5}{4} \cdot \binom{4}{2} \cdot 4! \cdot 6!$. By the same reasoning, $|A_5| = 5 \cdot 4! \cdot 6!$. By the sum rule, the event $A_1 \cup A_2 \cup A_3$ has size $45 \cdot 4! \cdot 6!$ and probability $45 \cdot 4! \cdot 6!/10! = 3/14 \approx 0.214$.

4. Use the inclusion-exclusion principle to answer the following questions:

(a) Is it possible to have 19 students and some number of clubs so that each club has at most 10 students, every two clubs have at least 4 students in common, but no three clubs have a student in common?

(b) In how many ways can you assign 20 different balls into 5 different bins so that there are no empty bins? (Hint: Calculate the size of the complement.)

Solution:

(a) Let $n$ be the number of clubs and $A_i$ be the set of students in club $i$. Since no three clubs have a student in common, by inclusion-exclusion we have

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|$$

The left side of the formula is the total number of students, which is 19. Since we have $n$ clubs and each club has at most 10 students, $\sum_{i=1}^n |A_i|$ is at most $10n$. Since there are $\binom{n}{2}$ pairs of clubs and every two clubs have at least 4 students, $\sum_{1 \leq i < j \leq n} |A_i \cap A_j|$ is at least $\binom{n}{2} \cdot 4$. Therefore

$$19 \leq 10n - 4\binom{n}{2} = 12n - 2n^2.$$
The function \( f(x) = 2x^2 - 12x + 19 \) does not have any real roots, so it is always positive. Therefore \( 12n - 2n^2 \) is always smaller than 19, so no such assignment of students to clubs is possible.

(b) Let \( A \) be the sets of assignments with at least one empty bin. Then \( A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \), where \( A_i \) is the set of assignment in which bin \( i \) is empty. By the product rule, \(|A_i| = 4^{20}\) for all \( i \), \(|A_i \cap A_j| = 3^{20}\) for all distinct \( i \) and \( j \), \(|A_i \cap A_j \cap A_k| = 2^{20}\) for all distinct \( i, j, k \), and so on. The inclusion-exclusion formula gives

\[
|A| = 5 \cdot 4^{20} - \binom{5}{2} \cdot 3^{20} + \binom{5}{3} \cdot 2^{20} - \binom{5}{4}
\]

and \(|A| = 5^{20} - |A| = 89,904,730,860,000.\)

5. Let \( Y \) be the set of arrangements of \( n \) stars and \( k \) bars in which no two bars are consecutive. For example when \( n = 6 \) and \( k = 3 \) then \(* \| * \| * \| * \| * \| * \) is in \( Y \), and \(* \| \| * \| * \| * \| * \) is not in \( Y \).

(a) List all the arrangements in \( Y \) when \( n = 6 \) and \( k = 3 \). How many are there?

(b) In general, what is the size of \( Y \)? Justify your answer.

(c) Let \( X \) be the set of all arrangements of \( n - k + 1 \) stars and \( k \) bars. For \( x \in X \), let \( f(x) \) be the sequence obtained by replacing every \(| \) in \( x \) by \(* \). For example, \( f(* \| \)) = * \| * \.

Prove that \( f \) is a bijection from \( X \) to \( Y \times \{*\} \).

Solution:

(a) These are the 35 arrangements; sorry there are more than we thought! \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \), \(| * \| * \| * \| * \| * \| * \).\)

(b) \(| Y | = \binom{n+1}{k} \). We insert \( k \) bars into the gaps of \( n \) stars and no two bars are inserted into the same gap since no two bars are consecutive. There are \( n + 1 \) gaps and thus, the size of \( Y \) is \( \binom{n+1}{k} \).

(c) \textit{Proof.} \( f \) is injective: Take any two different inputs from \( X \). Let \( i \) be the first position in which they differ. Then the portions of the output string corresponding to the first \( i - 1 \) input bits will be the same, but will differ at the next position, so the corresponding outputs must be different.

\( f \) is surjective: In every bit sequence \( y \) in \( Y \), every \(| \) is followed by a \(* \). Group this sequence into blocks of \(| \) and \(* \) and \(| \). If we replace each \(| \) by \(| \), we get a sequence \( x \) such that \( f(x) = y \), so \( f \) is surjective.

6. In this question you will investigate the best possible advantage for the second player in the game of intransitive dice. Given a set \( D \) of three dice, Alice chooses a die, Bob chooses one of the remaining two dice, then they each toss their die and the higher number wins. Let \( p(D) \) be the probability that Bob wins, assuming they both choose their die optimally.

(a) Calculate the value of \( p(D) \) for the following three dice:

Die A: 3, 3, 3, 3, 3, 3

Die B: 2, 2, 2, 5, 5, 5

Die C: 1, 4, 4, 4, 4, 4.
(b) Let $D = \{A, B, C\}$ be any set of three $n$-sided dice. Consider the experiment in which all three dice are tossed. Describe the sample space and the probabilities of all possible outcomes.

(c) If the face values of the outcome are $(a, b, c)$, let $E_{AB}$ be the event $a > b$, $E_{ABC}$ be the event $a > b > c$, and so on. Assume the dice have disjoint sets of face values. Show that

$$\Pr[E_{AB}] + \Pr[E_{BC}] = 1 + \Pr[E_{ABC}] - \Pr[E_{CBA}].$$

(d) Use part (c) to show that for every $D$, $p(D) \leq 2/3$.

(e) (Extra credit) Can you find a set of dice (no restriction on the number of faces) for which $p(D)$ is larger than the value you calculated in part (a)?

**Solution:**

(a) Out of the 36 possible outcomes for each pair of dice, there are 21 in which die A beats die B, 21 in which die B beats die C, and 25 in which die C beats die A. So Bob can always choose a die that beats Alice’s die with probability at least $21/36 = 7/12$.

(b) The sample space consists of all triples $(i, j, k)$, where $1 \leq i, j, k \leq n$, representing the face of die $A$, $B$, and $C$, respectively. All $n^3$ outcomes are equally likely, so each has probability $1/n^3$. The event that die $A$ beats die $B$ consists of those triples in which the value on face $i$ is larger than the value on face $j$, and so on.

(c) By the inclusion-exclusion principle for probabilities,

$$\Pr[E_{AB} \cup E_{BC}] = \Pr[E_{AB}] + \Pr[E_{BC}] - \Pr[E_{AB} \cap E_{BC}].$$

The event $E_{AB} \cap E_{BC}$ consists of those outcomes in which die A beats die B and die B beats die C, namely the event $E_{ABC}$. The event $E_{AB} \cup E_{BC}$ consists of those outcomes in which die A beats die B or die B beats die C. Its complement is the event in which die B beats die A and die C beats die B, namely the event $E_{CBA}$. Therefore $\Pr[E_{AB} \cup E_{BC}] = 1 - \Pr[E_{CBA}]$. Therefore

$$1 - \Pr[E_{CBA}] = \Pr[E_{AB}] + \Pr[E_{BC}] - \Pr[E_{ABC}].$$

The desired identity is obtained by rearranging terms.

(d) We prove this by contradiction. Assume that there exist a $D$ such that $p(D) > 2/3$. We consider two cases.

If $\Pr[E_{AB}] > 2/3$, then $\Pr[E_{CA}]$ must also be greater than $2/3$; if not, then Alice could win with probability at least $1/3$ by playing die A. By the same reasoning $\Pr[E_{BC}]$ must also be greater than $2/3$. By the inequality in part (b), it follows that all of $\Pr[E_{ABC}]$, $\Pr[E_{BCA}]$, $\Pr[E_{CAB}]$ are greater than $1/3$. Since the events $E_{ABC}$, $E_{BCA}$, and $E_{CAB}$ are disjoint, it follows that $\Pr[E_{ABC} \cup E_{BCA} \cup E_{CAB}] > 1$, a contradiction.

If $\Pr[E_{AB}] < 2/3$, then $\Pr[E_{BA}] \geq 1/3$, so $\Pr[E_{CB}]$ must be greater than $2/3$; if not Alice could win with probability at least $2/3$ by playing die $B$. By the same reasoning as in case one we can conclude that $\Pr[E_{CB}]$, $\Pr[E_{BA}]$, $\Pr[E_{AC}]$ are all greater than $2/3$ and obtain a contradiction by the same argument.

(e) It is possible, but we’ll leave this for some other time.