1. Find exact closed-form solutions to the following recurrences.

(a) \( T(n) = 3T(n/2) + n, T(1) = 1, \) where \( n \) is a power of 2.

(b) \( F(n) = \frac{1}{4} F(n-1) + n, F(0) = 0. \)

(c) \( f(n) = 8f(n-1) - 15f(n-2), f(0) = 0, f(1) = 1. \)

(d) \( f(n) = f(n-1) + f(n-2) + 1, f(0) = 0, f(1) = 1. \)

Solution:

(a) We try to guess a solution by unwinding the formula for \( T(n): \)

\[
T(n) = 3T\left(\frac{n}{2}\right) + n
\]

\[
= 3\left(3T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n = 3^2T\left(\frac{n}{2^2}\right) + 3 \cdot \frac{n}{2} + n
\]

\[
= 3^2\left(3T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + 3 \cdot \frac{n}{2} + n = 3^3T\left(\frac{n}{2^3}\right) + 3^2 \cdot \frac{n}{2^2} + 3 \cdot \frac{n}{2} + n
\]

Continuing in this manner suggests the guess

\[
T(n) = 3^{\log_2 n}T(1) + 3^{\log_2 n - 1} \cdot \frac{n}{2^\log_2 n} + \cdots + n.
\]

By \( T(1) = 1 \) and geometric sum formula, we can write

\[
T(n) = 3^{\log_2 n} + \frac{(3/2)^{\log_2 n} - 1}{(3/2) - 1} \cdot n = 3 \cdot 3^{\log_2 n} - 2n.
\]

If we write \( m = \log_2 n \), then \( T(2^m) = 3^{m+1} - 2^{m+1} \). We prove this is correct by induction on \( m \). When \( m = 0 \), both \( T(2^m) \) and \( 3^{m+1} - 2^{m+1} \) are one. Now assume \( T(2^m) = 3^{m+1} - 2^{m+1} \) for some \( m \). Then

\[
T(2^{m+1}) = 3T(2^m) + 2^{m+1} = 3 \cdot (3^{m+1} - 2^{m+1}) + 2^{m+1} = 3^{m+2} - 2^{m+2}
\]

as it should be.

(b) We try to guess a solution by unwinding the formula for \( F(n): \)

\[
F(n) = \frac{1}{3} F(n-1) + n
\]

\[
= \frac{1}{3} \left[ \frac{1}{3} F(n-2) + (n-1) \right] + n = \frac{1}{3^2} F(n-2) + \frac{1}{3} (n-1) + n
\]

\[
= \frac{1}{3^2} \left[ \frac{1}{3} F(n-3) + (n-2) \right] + \frac{1}{3} (n-1) + n = \frac{1}{3^3} F(n-3) + \frac{1}{3^2} (n-2) + \frac{1}{3} (n-1) + n.
\]

Continuing in this manner suggests the guess

\[
F(n) = \frac{1}{3^n} F(0) + \frac{1}{3^{n-1}} + \frac{2}{3^{n-2}} + \cdots + n = \frac{1}{3^{n-1}} \left( 1 + 2 \cdot 3 + \cdots + n \cdot 3^{n-1} \right).
\]

To evaluate the last sum, recall that in question 2d of Homework 4 we showed that for every \( x \neq 1, \)

\[
1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{(n+1)x^n(x-1) - (x^{n+1}-1)}{(x-1)^2}
\]
In particular, evaluating this expression at $x = 3$ gives

$$1 + 2 \cdot 3 + \cdots + n \cdot 3^{n-1} = \frac{(2n - 1) \cdot 3^n + 1}{4}.$$  

From here we obtain the guess

$$F(n) = \frac{(2n - 1) \cdot 3^n + 1}{4 \cdot 3^{n-1}} = \frac{3n}{2} - \frac{3}{4} + \frac{3}{4} \cdot 3^{-n}.$$  

We prove this is correct by induction on $n$. When $n = 0$, both sides are zero. Now assume $F(n) = 3n/2 - 3/4 + (3/4) \cdot 3^{-n}$ for some $n$. Then

$$F(n + 1) = \frac{1}{3} F(n) + (n + 1)$$

$$= \frac{1}{3} \left( \frac{3n}{2} - \frac{3}{4} + \frac{3}{4} \cdot 3^{-n} \right) + (n + 1)$$

$$= \frac{3(n + 1)}{2} - \frac{3}{4} + \frac{3}{4} \cdot 3^{-(n+1)}$$

as it should be.

(c) This is a homogeneous linear recurrence, so we guess a solution of the form $f(n) = x^n$ for some nonzero $x$. If our guess is correct, $x^n$ must equal $8x^{n-1} - 15x^{n-2}$ for all $n$, from where $x^2 = 8x - 15$. This quadratic equation has the two solutions $x_1 = 3$ and $x_2 = 5$. Any linear combination of $x_1^n$ and $x_2^n$ also satisfies the recurrence. We look for a linear combination $f(n) = sx_1^n + tx_2^n$ that satisfies the additional requirements $f(0) = 0$ and $f(1) = 1$:

$$0 = f(0) = s + t$$
$$1 = f(1) = sx_1 + tx_2 = 3s + 5t.$$  

The unique solution to this system is $s = -1/2$ and $t = 1/2$. Therefore

$$f(n) = -\frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot 5^n = \frac{5^n - 3^n}{2}$$

is the solution to our recurrence.

(d) We first homogenize the recurrence. By adding one to both sides we can write

$$f(n) + 1 = (f(n - 1) + 1) + (f(n - 2) + 1).$$

The function $g(n) = f(n) + 1$ then satisfies the recurrence $g(n) = g(n - 1) + g(n - 2)$ with initial conditions $g(0) = 1$ and $g(1) = 2$. This is the same as the recurrence from Section 4 of Lecture 8, but with different initial conditions. We can solve it using the same method. Alternatively we can reason like this. If we define $g(-1) = 1$ then the recurrence is still satisfied for all $n \geq 1$. Then the function $h(n) = g(n - 1)$ satisfies the same recurrence but with initial conditions $h(0) = g(-1) = 1$ and $h(1) = g(0) = 1$. This is exactly the same as the recurrence in Lecture 8, so

$$h(n) = \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} + \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1},$$

from where

$$g(n) = h(n - 1) = \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} + \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2},$$

and finally

$$f(n) = g(n) - 1 = \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} + \frac{1}{\sqrt{5}} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^{n+2} - 1.$$
2. A password consists of the digits 0 to 9 and the special symbols * and #. How many 6 to 8-symbol passwords are there if

(a) all symbols must be different?
(b) the password must have at least one digit?
(c) all the digits in the password must be the same?
(d) no two special symbols are consecutive? (Hint: Write a recurrence.)

Solution:

(a) We write the set of passwords as a disjoint union of the sets $E_6$, $E_7$, and $E_8$ consisting of those passwords with exactly 6, 7, and 8 symbols. By the generalized product rule, $|E_6| = 12 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$, $|E_7| = 12 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$, and $|E_8| = 12 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$, so the total number of passwords is the sum of these three numbers, namely 24,615,360.

(b) For 6-symbol passwords, there are $12^6$ possible passwords, and $2^6$ of them are without any digit. So, there are $12^6 - 2^6 = 2,985,920$ 6-symbol passwords with at least one digit. Similarly, there are $12^7 - 2^7 = 35,831,680$ 7-symbol passwords with at least one digit, and $12^8 - 2^8 = 429,981,440$ 8-symbol passwords with at least one digit. In total, there are 2,985,920+35,831,680+429,981,440 = 468,799,040 6 to 8-symbol passwords with at least one digit.

(c) We write the set of passwords as a disjoint union of the sets $N$ and $A_0$, $A_1$, up to $A_9$, where the set $N$ consists of all passwords without any digits, and $A_i$ is the set of passwords that have at least one digit and all digits in the password are equal to $i$. The set $N$ consists of all the passwords made up only of #s and *s, so by the sum and product rules its size is $|N| = 3^6 + 3^7 + 3^8$. The set $A_i \cup N$ consists of those passwords made up of the symbols #, *, and the digit $i$ only, so $|A_i \cup N| = 3^6 + 3^7 + 3^8$. By the sum rule, $|A_i| = |A_i \cup N| - |N|$ and so the total number of passwords is
\[
|N \cup A_0 \cup \ldots \cup A_9| = |N| + |A_0| + \cdots + |A_9|
= |N| + (|A_0 \cup N| - |N|) + \cdots + (|A_9 \cup N| - |N|)
= 10 \cdot (3^6 + 3^7 + 3^8) - 9 \cdot (2^6 + 2^7 + 2^8)
= 90738.
\]

(d) Let $f(n)$ be the number of $n$-symbol passwords in which no two special symbols are consecutive. Consider an $n$-symbol password in which no two special symbols are consecutive. If the first symbol is a digit, then it can be followed by any $(n-1)$-symbol password in which no two special symbols are consecutive. There are $10 \cdot f(n-1)$ such passwords. If the first symbol is a special symbol, then the second symbol must be a digit, followed by any $(n-2)$-symbol password in which no two special symbols are consecutive. There are $2 \cdot 10 \cdot f(n-2)$ such passwords. So $f(n) = 10f(n-1) + 20f(n-2)$. The initial conditions are $f(0) = 1$ and $f(1) = 12$, from where we obtain that $f(2) = 140$, $f(3) = 1640$, $f(4) = 19200$, $f(5) = 224800$, $f(6) = 2632000$, $f(7) = 30816000$, and $f(8) = 360800000$. The desired number of passwords is $f(6) + f(7) + f(8) = 2,632,000 + 30,816,000 + 360,800,000 = 394,248,000$.

3. Use the pigeonhole principle to prove the following propositions.

(a) Among the 20000 students at CUHK, there are at least 1100 that all live in the same district. (Hong Kong has 18 districts.)

(b) You throw three six-sided dice repeatedly. The score of each throw is the sum of the face values of the three dice. Among 50 repetitions, at least four have the same score.
(c) In every graph there are two distinct vertices of equal degrees.

Solution:

(a) Consider the function that assigns to each students the distinct where he/she lives. This function has a domain of size 20000 and a range of size 18. Because $20000 > 18 \cdot 1100$, by the generalized pigeonhole principle, there must exists a distinct in which at least 1100 students live.

(b) There are 16 possible scores of a throw of three six-sided dice: 3, 4, 5, 6, 7, \cdots, 18. Consider the function that assigns to each of the 50 throws the score of that throw. This function has a domain of size 50 and a range of size 16. Since $50 > 16 \cdot 3$, by the generalized pigeonhole principle there must be at least four repetitions having the same score.

(c) Let $n$ be the number of vertices and consider the function deg that assigns the degree to every vertex. The degree is a number between zero and $n-1$. We consider two cases. If there is no vertex of degree zero, then deg has a domain of size $n$ and a range of size $1$ so by the pigeonhole principle two vertices must have the same degree. Otherwise there is a vertex $v$ of degree zero. Then the degree of any other vertex $w$ is at most $n-2$ because there cannot be an edge between $w$ and $v$. So the range of deg has size at most $n-2$ again, and by the pigeonhole principle two vertices must have the same degree.

4. For each of the following pairs of functions, say whether (i) $g$ is $o(f)$, (ii) $g$ is $\Theta(f)$, or (iii) $f$ is $o(g)$. Justify your answer.

(a) $f(n) = e^n$, $g(n) = n^e$.
(b) $f(n) = n^n$, $g(n) = 2n^2$.
(c) $f(n) = \sqrt{n}$, $g(n) = 2\log n$.
(d) $f(n) = f(\lfloor n/5 \rfloor) + 2 \cdot f(\lfloor 2n/5 \rfloor) + n^2$, $g(n) = 1 \log 1 + 2 \log 2 + \cdots + n \log n$. ($\lfloor x \rfloor$ is the largest integer not exceeding $x$.)

Solution:

(a) By Corollary 5 in Lecture 7, $g$ is $o(f)$.

(b) $f$ is $o(g)$. We can write $f(n) = 2^{n \log n}$. Then $f(n)/g(n) = 2^{n \log n - n^2} = (2^n)^{-n + \log n}$. As $n$ grows the value $-n + \log n$ tends to minus infinity, so the ratio $f(n)/g(n)$ tends to zero.

(c) $g$ is $o(f)$. We can write $f(n) = 2^{\frac{1}{2} \log n}$. Then $g(n)/f(n) = 2^{\sqrt{\log n} - \frac{1}{2} \log n}$. The difference $\frac{1}{2} \log n - \sqrt{\log n}$ is $\Omega(\log n)$, so the exponent tends to minus infinity and the ratio tends to zero as $n$ grows.

(d) $f$ is $o(g)$ By the formula in Section 5 of Lecture 7, $f(n)$ is $\Theta(n^p(1 + \int_1^n u^{2-p} du))$, where $p$ is the unique integer satisfying $(1/5)^p + 2 \cdot (2/5)^p = 1$. Since $p = 1$, the integral of interest evaluates to $n$ and we obtain that $f(n)$ is $\Theta(n^2)$.

The growth rate of $g$ can be estimated by the integral method but this is not necessary. It is enough to argue that $g$ grows at a rate asymptotically faster than $n^2$. This is true because each term of $g$ occurring in the second half of the summation is larger than $(n/2) \log(n/2)$ and there are at least $n/2 - 1$ such terms. Therefore

$$g(n) \geq \left(\frac{n}{2} - 1\right) \cdot \frac{n}{2} \log(n/2) = \Omega(n^2 \log n).$$

Since $n^2$ is $o(n^2 \log n)$ it follows that $f$ is $o(g)$. 
5. DNA (Deoxyribonucleic acid) is a molecule that carries the genetic instructions for all known organisms and many viruses. It consists of a chain of bases. In DNA chain, there are four types of bases: A, C, G, T. For example, a DNA chain of length 10 can be ACGTACGTAT.

(a) Let $g(n)$ be the number of configurations of a DNA chain of length $n$, in which no two T are consecutive and no two G are consecutive. Write a recurrence for $g(n)$.

(b) Solve the recurrence from part (a).

(c) Let $h(n)$ be the number of configurations of a DNA chain of length $n$, in which no two T are consecutive, no two G are consecutive, and T, G are not next to each other. Write a recurrence for $h(n)$.

(d) Solve the recurrence from part (c).

Solution:

(a) Let $A(n)$, $C(n)$, $T(n)$ and $G(n)$ be the numbers of configurations of DNA chains of length $n$ in which no two T are consecutive and no two G are consecutive, starting with A, C, G and T, respectively. Now, we know that $g(n) = A(n) + C(n) + G(n) + T(n)$.

Consider a DNA chain of length $n$. If the first base is A, then it can be followed by any DNA chain of length $n - 1$. If the first base is C, then it can be followed by any DNA chain of length $n - 1$. If the first base is G, then it can be followed by any DNA chain of length $n - 1$ starting with A, C or T. If the first base is T, then it can be followed by any DNA chain of length $n - 1$ starting with A, C or G. Therefore, we have

\[
\begin{align*}
A(n) &= g(n - 1) \\
C(n) &= g(n - 1) \\
G(n) &= A(n - 1) + C(n - 1) + T(n - 1) = 2g(n - 2) + T(n - 1) \\
T(n) &= A(n - 1) + C(n - 1) + G(n - 1) = 2g(n - 2) + G(n - 1)
\end{align*}
\]

By summing above equations, we have

\[
g(n) = A(n) + C(n) + G(n) + T(n) = 2g(n - 1) + 4g(n - 2) + (T(n - 1) + G(n - 1))
\]

\[
= 2g(n - 1) + 4g(n - 2) + (g(n - 1) - A(n - 1) - C(n - 1))
\]

\[
= 2g(n - 1) + 4g(n - 2) + g(n - 1) - g(n - 2) - g(n - 2)
\]

\[
= 3g(n - 1) + 2g(n - 2)
\]

(b) This is a homogeneous linear recurrence, so we guess a solution of the form $g(n) = x^n$ for some nonzero $x$. If our guess is correct, $x^n$ must equal $3x^{n-1} + 2x^{n-2}$ for all $n$, from where $x^2 = 3x + 2$. This quadratic equation has the two solutions $x_1 = (3 - \sqrt{17})/2$ and $x_2 = (3 + \sqrt{17})/2$. Any linear combination of $x_1^n$ and $x_2^n$ also satisfies the recurrence. We look for a linear combination $g(n) = sx_1^n + tx_2^n$ that satisfies the additional requirements $g(1) = 4$ and $g(2) = 14$:

\[
4 = g(1) = \frac{3 - \sqrt{17}}{2} s + \frac{3 - \sqrt{17}}{2} t
\]

\[
14 = g(2) = \left(\frac{3 - \sqrt{17}}{2}\right)^2 s + \left(\frac{3 - \sqrt{17}}{2}\right)^2 t.
\]

The unique solution to this system is $s = 1/2 - 5/(2\sqrt{17})$ and $t = 1/2 + 5/(2\sqrt{17})$. Therefore

\[
g(n) = \left(\frac{1}{2} - \frac{5}{5\sqrt{17}}\right) \left(\frac{3 - \sqrt{17}}{2}\right)^n + \left(\frac{1}{2} + \frac{5}{5\sqrt{17}}\right) \left(\frac{3 + \sqrt{17}}{2}\right)^n
\]
is the solution to our recurrence.

(c) Consider a DNA chain of length \(n\). If the first base is \(A\), then it can be followed by any DNA chain of length \(n-1\). If the first base is \(C\), then it can be followed by any DNA chain of length \(n-1\). If the first base is \(G\), then the second base must be either \(A\) or \(C\), followed by any DNA chain of length \(n-2\). If the first base is \(T\), then the second base must be either \(A\) or \(C\), followed by any DNA chain of length \(n-2\). Therefore, we have \(h(n) = h(n-1) + h(n-1) + h(n-2) + h(n-2) = 2h(n-1) + 2h(n-2)\).

(d) This is a homogeneous linear recurrence, so we guess a solution of the form \(h(n) = x^n\) for some nonzero \(x\). If our guess is correct, \(x^n\) must equal \(2x^{n-1} + 2x^{n-2}\) for all \(n\), from where \(x^2 = 2x + 2\). This quadratic equation has the two solutions \(x_1 = 1 - \sqrt{3}\) and \(x_2 = 1 + \sqrt{3}\). Any linear combination of \(x_1^n\) and \(x_2^n\) also satisfies the recurrence. We look for a a linear combination \(h(n) = sx_1^n + tx_2^n\) that satisfies the additional requirements \(h(1) = 4\) and \(h(2) = 13:\)

\[
4 = h(1) = (1 - \sqrt{3})s + (1 + \sqrt{3})t
\]
\[
13 = h(2) = (1 - \sqrt{3})^2s + (1 + \sqrt{3})^2t.
\]

The unique solution to this system is \(s = (5 - \sqrt{3})/4\) and \(t = (5 + \sqrt{3})/4\). Therefore

\[
h(n) = \frac{5 - \sqrt{3}}{4}(1 - \sqrt{3})^n + \frac{5 + \sqrt{3}}{4}(1 + \sqrt{3})^n
\]

is the solution to our recurrence.

6. A pair of permutations of \(\{1, \ldots, n\}\) is a special pair if there is some position in which they differ by exactly one. For example, \(\{(3,1,2,4),(1,4,3,2)\}\) (when \(n = 4\)) is a special pair because they differ by exactly one in the third position, but \(\{(1,2,3,4),(1,4,3,2)\}\) is not a special pair. A set of permutations \(S_n\) of \(\{1, \ldots, n\}\) is a special set if every two permutations within \(S_n\) are a special pair.

(a) Show that when \(n = 3\), there exists a special set of size 3 but no special set of size 4.

(b) Show that if \(S_n\) is a special set, the function \(f: S_n \rightarrow \{0, 1\}^n\) given by \(f((x_1, x_2, \ldots, x_n)) = (x_1 \mod 2, x_2 \mod 2, \ldots, x_n \mod 2)\) is injective.

(c) Use part (b) to show that if \(S_n\) is a special set then \(|S_n| \leq 2^n\).

(d) Define the sets \(S_1, S_2, \ldots\) recursively by the formula \(S_n = A_n \cup B_n \cup C_n\) where

\[
A_n = \{(n, n-1, x_1, x_2, \ldots, x_{n-2}): (x_1, x_2, \ldots, x_{n-2}) \in S_{n-2}\},
\]
\[
B_n = \{(x_1, n-1, x_2, \ldots, x_{n-2}): (x_1, x_2, \ldots, x_{n-2}) \in S_{n-2}\},
\]
\[
C_n = \{(n-1, x_1, n, x_2, \ldots, x_{n-2}): (x_1, x_2, \ldots, x_{n-2}) \in S_{n-2}\}.
\]

with \(S_1 = \{(1)\}\) and \(S_2 = \{(1,2), (2,1)\}\). Show that \(S_n\) is a special set for all \(n\).

(e) Give a formula for the size of the sets \(S_n\) from part (d).

(f) (Extra credit) For \(n = 8\), can you find a special set larger than \(S_8\) from part (d)?

Solution:

(a) The set \(\{(1,2,3),(2,3,1),(3,1,2)\}\) is a special set of size 3. To show there is no special set of size 4, we first partition the set of all permutations of \(\{1,2,3\}\) into the pairs \(N_1 = \{(2,1,3),(2,3,1)\}, N_2 = \{(1,2,3),(1,3,2)\}\), and \(N_3 = \{(1,3,2),(3,1,2)\}\). None of these pairs is special. Therefore a special set must contain at most one permutation from each of these sets, so it can have size at most 3.
(b) Let \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) be any two special permutations in \(S_n\). Then \(x_i - y_i = 1\) for some coordinate \(i\). It follows that \(x_i \mod 2 - y_i \mod 2 = 1\), so \(x_i \mod 2 \neq y_i \mod 2\). Therefore \(f(x_1, \ldots, x_n)\) and \(f(y_1, \ldots, y_n)\) differ in the \(i\)-th coordinate so they are not equal.

(c) Suppose \(S_n\) is a special set. The function \(f : S_n \rightarrow \{0, 1\}^n\) defined in part (b) is injective, so \(|S_n| \leq |\{0, 1\}^n| = 2^n\).

(d) We prove it by strong induction on \(n\). \(S_1\) has only one element, so \(S_1\) is a special set. \((1, 2), (2, 1)\) is a special pair, so \(S_2\) is also a special set. Now assume \(n \geq 3\) and \(S_{n-2}\) is special. We will prove that \(S_n\) is also special. Let \(x\) and \(y\) be any two distinct elements in \(S_n\). We consider two cases. If \(x, y \in A_n\) or \(x, y \in B_n\) or \(x, y \in C_n\) then the permutations \(x'\) and \(y'\) obtained by removing entries \(n\) and \(n-1\) from \(x\) and \(y\) belong to \(S_{n-2}\), so the pair \(\{x', y'\}\) is special (namely, there is a coordinate in which they differ by one). Permutations \(x\) and \(y\) are obtained by extending \(x'\) and \(y'\) by two entries in the same position, so pair \(\{x, y\}\) must also be special (they differ by one in the same coordinate). If, on the other hand, \(x\) and \(y\) belong to two different sets among \(A_n\), \(B_n\), and \(C_n\), then there exists a coordinate \(i\) among the first three in which \(\{x_i, y_i\} = \{n, n-1\}\), so \(x\) and \(y\) differ by one in this coordinate. Therefore the pair is special.

(e) From part (d), we know that \(|S_1| = 1\), \(|S_2| = 2\) and \(|S_n| = 3|S_{n-2}|\) for all \(n > 2\), so

\[
S_n = \begin{cases} 
2 \cdot 3^{n/2-1}, & \text{if } n \text{ is even,} \\
3^{(n-1)/2}, & \text{if } n \text{ is odd.}
\end{cases}
\]