Questions

1. Find exact closed form expressions for the following sums. Explain how you discovered the expression and prove that it is correct.

(a) \( 1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 \).

**Solution:** \( 1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = A - B \), where

\[
A = 1^2 + 2^2 + \cdots + (2n + 1)^2 = \frac{1}{3}(2n + 1)^3 + \frac{1}{2}(2n + 1)^2 + \frac{1}{6}(2n + 1)
\]

by Theorem 1 from Lecture 7

\[
B = 2^2 + 4^2 + \cdots + (2n)^2 = 4(1^2 + 2^2 + \cdots + n^2) = \frac{4}{3}n^3 + \frac{4}{2}n^2 + \frac{4}{6}n
\]

by the same theorem. After simplifying the expression \( A - B \) we get that

\[
1^2 + 3^2 + 5^2 + \cdots + (2n + 1)^2 = \frac{4}{3}n^3 + 4n^2 + \frac{11}{3}n + 1.
\]

(b) \( (2^0 + \cdots + 2^n) + (2^1 + \cdots + 2^{n+1}) + \cdots + (2^n + \cdots + 2^{2n}) \).

**Solution:** The \((i + 1)\)st term in this expression is

\[
2^i + 2^{i+1} + \cdots + 2^{i+n} = 2^i(1 + \cdots + 2^n) = 2^i(2^{n+1} - 1).
\]

The sum of these terms as \( i \) ranges from 0 to \( n \) is

\[
(2^0 + 2^1 + \cdots + 2^n)(2^{n+1} - 1) = (2^{n+1} - 1)^2.
\]

2. Show the following inequalities.

(a) \( 2\sqrt{n+1} - 2 \leq 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n+1} - 1 - \frac{1}{\sqrt{n+1}} \).

**Solution:** The value of the sum from 1 to \( n \) equals the area under the first \( n \) bars.

This area is at least the integral of the function \( 1/\sqrt{x} = x^{-1/2} \) from 1 to \( n + 1 \):

\[
1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \int_1^{n+1} x^{-1/2} \, dx = 2x^{1/2}\big|_1^{n+1} = 2\sqrt{n+1} - 2.
\]

To obtain the other side of the inequality we subtract the area of the grey bars, which is \( 1 - 1/\sqrt{n+1} \).
(b) \( n^n \leq 1^1 + 2^2 + \cdots + n^n \leq (1 + \frac{1}{n-1})n^n \) (for \( n \geq 2 \)).

**Solution:** The left inequality is clearly true since the sum of the first \( n-1 \) terms is nonnegative. For the right one, we can write
\[
1^1 + 2^2 + \cdots + n^n \leq n^1 + n^2 + \cdots + n^n \\
= n \cdot (1 + n + \cdots + n^{n-1}) \\
\leq n \cdot \frac{n^n - 1}{n - 1} \\
= \frac{n}{n-1}(n^n - 1) \\
= \left(1 - \frac{1}{n-1}\right)(n^n - 1)
\]
by the geometric sum formula from Lecture 7. This is slightly smaller than the required bound.

(c) \(|H(n) - \ln n| = O(1)\), where \( H(n) \) is the \( n \)th harmonic number.

(\textbf{Hint:} You need to show that \( H(n) \geq \ln n - O(1) \) and that \( H(n) \leq \ln n + O(1) \).)

**Solution:** In Lecture 7 we showed that \( H(n) \geq \ln(n + 1) \), so \( H(n) \geq \ln n \). We also showed that
\[
H(n) \leq \ln(n + 1) + 1 - \frac{1}{n + 1} \leq \ln(n + 1) + 1
\]
so when \( n \geq 1 \)
\[
H(n) - \ln n \leq \ln(n + 1) - \ln n + 1 \leq \ln(2n) - \ln n + 1 = \ln 2 + 1.
\]

3. Sort the following functions in increasing order of asymptotic growth. (For example, if you are given the functions \( n^2, n, \) and \( 2^n \), the sorted list would be \( n, n^2, 2^n \).) Show that for every pair of consecutive functions \( f, g \) in your list, \( f \) is \( o(g) \).

(a) \( n^\log n, 2^n, n^n, 2^{\log \log n}, 2^n, e^n, n, 2^n, (\log n)^{(\log n)^{(\log n)}} \), \( n^{(\log n)^{\log \log n}} \).

**Solution:** The sorted list is
\[
2^{\log \log n}, n, n^{\log n}, n^{(\log n)^{\log \log n}}, 2^n, n^n, 2^n, (\log n)^{(\log n)^{(\log n)}} , e^n, 2^n.
\]
To show correctness, we repeatedly use the following claim:

**Claim 1** If \( c(n) \geq 2 \) for sufficiently large \( n \), \( f \) is \( o(g) \), and \( \lim_{n \to \infty} g(n) = \infty \), then \( c(n)^f \) is \( o(c(n)^g) \).

Below, we abbreviate the conclusion as \( c(n)^{o(g)} = o(c(n)^g) \).

**Proof:** Since \( f \) is \( o(g) \), \( f(n) \leq \frac{1}{2}g(n) \) for sufficiently large \( n \). Then for \( n \) sufficiently large,
\[
\frac{c(n)^{f(n)}}{c(n)^{g(n)}} = c(n)^{f(n)-g(n)} \leq c(n)^{-\frac{1}{2}g(n)} \leq 2^{-\frac{1}{2}g(n)}
\]
which tends to zero as \( n \) gets large.

We now show the ordering of the sequence is correct.
4. Find exact closed-form solutions to the following recurrences.

(a) \( g(0) = 0, g(1) = 3, g(n) = g(n - 1) + 2g(n - 2) \) for \( n \geq 2 \).

**Solution:** The solution is \( g(n) = (-1)^{n+1} + 2^{n+1} \). Here is how I found it. First I disregard the base conditions and guess a solution of the form \( g(n) = c^n \). Then \( c \) must satisfy the equation
\[
c^2 = c + 2
\]
which has solutions \( c_1 = -1, c_2 = 2 \). Therefore all functions of the form \( g(n) = a(-1)^n + b2^n \) solve the recurrence. Taking the initial conditions into account, I get \( a + b = g(0) = 0 \) and \(-a + 2b = g(1) = 3\), which solves to \( a = -1, b = 1 \).

(b) The lower function \( T(n) \) is given by the recursive formula \( T(1) = 2 \) and \( T(n + 1) = 2T(n) \) for \( n \geq 1 \). Where do the functions \( T(n) \) and \( T(\log n) \) (assuming \( n \) is a power of two) fit into the list in part (a)?

**Solution:** They come at the very end of the list in the order \( T(\log n), T(n) \). To show this, first we prove by induction that \( T(n) \geq 2^{n-1} \geq n \) for all \( n \): This is true for \( n = 1 \), and assuming it is true for a given \( n \geq 1 \),
\[
T(n + 1) = 2T(n) \geq 2^n = 2 \cdot 2^{n-1} \geq 2 \cdot n \geq n + 1,
\]
it is also true for \( n + 1 \).

To show \( 2^n = o(T(\log n)) \), we can write
\[
2^n = 2^{2n \log e} = 2^{2n \log n + \log \log e} = 2^{2n \log (\log n + \log \log e)}.
\]

Now \( \log (\log n + \log e) \) is \( o(\log n - 4) \). Since \( \log n - 4 \leq T(\log n - 4) \), we get that
\[
2^{2n \log (\log n + \log e)} = 2^{2n \log (T(\log n - 4))} = 2^{2n \log (2^{T(\log n - 4)})} = 2^{2n \log (2^{T(n-3)})} = 2^{2n \log (T(n-2))} = 2^{2n \log (T(n-1))} = o(T(\log n)).
\]

For the second inequality, since \( T \) is an increasing function, \( T(n - 1) = \log T(n) = o(T(n)) \), and so \( T(\log n) \leq T(n - 1) = o(T(n)) \).

4. Find exact closed-form solutions to the following recurrences.

(a) \( n^v \)
(b) \( T(n) = 3T(n/3) + 3n, \ T(1) = 0, \ n \) is a power of 3.

**Solution:** For a sufficiently large \( n \), we can write

\[
T(n) = 3T(n/3) + 3n \\
= 3(3T(n/3^2) + 3(n/3)) + 3n \\
= 3^2(3T(n/3^3) + 3n/3^3) + 2 \cdot 3n = 3^3T(n/3^3) + 3 \cdot 3n.
\]

After \( \log_3 n - 1 \) steps, we get an expression of the form \( 3^{\log_3 n}T(1) + \log_3 n \cdot 3n = 3n \log_3 n \).

Let’s verify that this works by strong induction on \( n \). When \( n = 1, 3n \log_3 n = 0 = T(1) \). Now assume \( n \) is a power of 3, \( n \geq 1, \) and \( T(k) = 3k \log_3 k \) for every \( k \) that is a power of 3 between 1 and \( n - 1 \). Then

\[
T(n) = 3T(n/3) + 3n = 3 \cdot 3(n/3) \log_3(n/3) + 3n = 3n(\log_3 n - 1) + 3n = 3n \log_3 n.
\]

We conclude that \( T(n) = 3n \log_3 n \) for every \( n \) that is a power of 3.

(c) \( F(n) = 4F(n/4) + n^2, \ F(1) = 1, \ n \) is a power of 4.

**Solution:** For a sufficiently large \( n \), we can write

\[
F(n) = 4F(n/4) + n^2 \\
= 4(4F(n/4^2) + (n/4)^2) + n^2 = 4^2F(n/4^2) + (1 + 1/4)n^2 \\
= 4^2(4F(n/4^3) + (n/4^2)^2) + (1 + 1/4)n^2 = 4^3F(n/4^3) + (1 + 1/4 + 1/4^2)n^2.
\]

After \( \log_4 n - 1 \) steps, we get an expression of the form

\[
4^{\log_4 n}F(1) + (1 + 1/4 + 1/4^2 + \cdots + 1/4^{\log_4 n-1})n^2 = n + \frac{1 - (1/4)^{\log_4 n}}{1 - 1/4} \cdot n^2 \\
= n + \frac{4}{3} \left(1 - \frac{1}{n}\right)n^2 \\
= \frac{4}{3}n^2 - \frac{1}{3}n.
\]

We show this guess is correct by strong induction on \( n \). For \( n = 1, (4/3)n^2 - (1/3)n = 1 = F(n) \). Let \( n \geq 1 \) be a power of 4 and assume that \( F(k) = (4/3)k^2 - (1/3)k \) for all smaller powers of 4. Then

\[
F(n) = 4F(n/4) + n^2 = 4(\frac{4}{3}(n/4)^2 - \frac{1}{3}(n/4)) + n^2 = \frac{4}{3}n^2 - \frac{1}{3}n.
\]
5. The Master Theorem states that the solution to the recurrence

$$T(n) = aT(n/b) + g(n),$$

where $a$ and $b$ are constants, has asymptotic behaviour

$$T(n) = \begin{cases} 
\Theta(n^{\log_b a}), & \text{if } g(n) = O(n^c) \text{ for some } c < \log_b a \\
\Theta(g(n)), & \text{if } g(n) = \Omega(n^c) \text{ for some } c > \log_b a.
\end{cases}$$

Use Theorem 5 from Lecture 8 (the Akra-Bazzi Theorem) to prove the Master Theorem. Assume that $|dg(x)/dx| = O(x^C)$ for some constant $C$.

**Solution:** The Akra-Bazzi theorem tells us that

$$T(n) = \Theta\left(n^p \left(1 + \int_1^n \frac{g(u)}{u^{p+1}} \, du\right)\right)$$

where $p$ is chosen so that $ab^p = 1$, that is $p = \log_b a$. If $g(n) = O(n^c)$ for some constant $c < p$ and sufficiently large $n$ then for $\varepsilon = p - c$

$$\int_1^n \frac{g(u)}{u^{p+1}} \, du = O\left(\int_1^n \frac{1}{u^{1+\varepsilon}} \, du\right) = O(1)$$

because the integral converges. It follows that $T(n) = \Theta(n^p(1 + O(1))) = \Theta(n^p)$.

I don’t know how to derive the second part without imposing additional constraints on the function $g$. You can check that it is correct for functions of the form $g(n) = n^c$ by calculating the integral.

6. You want to move the Towers of Hanoi, but now you have four poles instead of three. The rules are the same: $n$ disks are initially stacked up from largest at the bottom to smallest on top on the leftmost pole. The objective is to move them to the rightmost pole one by one so that at no point does a larger disk cover a smaller one.

Consider the following strategy: If $n \leq 10$, ignore one of the poles and apply the solution from class for three poles. If $n > 10$, recursively move the top $n - 10$ disks to the second pole, stack up the bottom 10 disks onto the last pole using the other three poles only, and then recursively move the $n - 10$ remaining disks from the second pole to the last pole.

Let $T(n)$ be the number of steps that it takes to move the whole stack of $n$ disks.

(a) Write a recurrence for $T(n)$. Explain why your recurrence is correct.

**Solution:** The number of moves the strategy makes for $n$ disks and 4 poles equals twice the number of moves for $n - 10$ disks and 4 poles, plus the number of moves for 10 disks and 3 poles which equals $2^{10} - 1 = 1023$. Therefore the recurrence is

$$T(n) = 2T(n - 10) + 1023$$

for $n > 10$ and $T(n) = O(1)$ for $n \leq 10$. 
(b) Show that the recurrence from part (a) satisfies \( T(n) = O(2^{n/10}) \).

**Solution:** For \( n \) sufficiently large,

\[
T(n) = 2T(n - 10) + 1023 = 2^2T(n - 2 \cdot 10) + 2 \cdot 1023 + 1023 = 2^3T(n - 3 \cdot 10) + (1 + 2 + 2^2)1023.
\]

After \([n/10]\) steps, we get

\[
T(n) = 2^{[n/10]}T(n - [n/10] \cdot 10) + (1 + 2 + \cdots + 2^{[n/10]}) \leq 2^{n/10}T(k) + (2^{[n/10]+1} - 1)
\]

for some \( k \) between 1 and 10. The last expression is \( O(2^{n/10}) \).

(c) (Extra credit) Can you come up with a different strategy in which \( 2^{O(\sqrt{n})} \) moves are sufficient?

**Solution:** When \( n \geq 2 \), recursively move the top \( n - [\sqrt{n}] \) disks to the second pole, stack up the bottom \( [\sqrt{n}] \) disks onto the last pole using the other three poles only, and then recursively move the \( n - [\sqrt{n}] \) remaining disks from the second pole to the last pole. This gives the recurrence

\[
T(n) = 2T(n - [\sqrt{n}]) + 2^{[\sqrt{n}]} - 1
\]

for the number of moves. Since \( T \) is an increasing function for every \( k \) in the range \( n/2 \leq k \leq n \), we can write

\[
T(k) = 2T(k - [\sqrt{k}]) + 2^{[\sqrt{k}]} - 1 \leq 2T(k - [\sqrt{n/2}]) + 2^{[\sqrt{n}]}.
\]

Therefore for \( n \) sufficiently large

\[
T(n) \leq 2T(n - [\sqrt{n/2}]) + 2^{[\sqrt{n}]}
\]

\[
\leq 2^2T(n - 2[\sqrt{n/2}]) + (1 + 2)2^{[\sqrt{n}]}
\]

\[
\leq 2^3T(n - 3[\sqrt{n/2}]) + (1 + 2 + 2^2)2^{[\sqrt{n}]}.
\]

Continuing for just enough steps \( t \) so that the argument of \( T(\cdot) \) drops below \( n/2 \), we get that

\[
T(n) \leq 2^tT([n/2]) + (2^t - 1)2^{[\sqrt{n}]}
\]

The value of \( t \) can be at most \( 2^{[\sqrt{n/2}]} \), so

\[
T(n) \leq 2^{[\sqrt{n/2}]}T([n/2]) + 2^{[\sqrt{n/2}]} \cdot 2^{[\sqrt{n}]} \leq 2^{[\sqrt{n}]}T([n/2]) + 2^{2[\sqrt{n}]}.\]

We can derive the inequality

\[
T(n) + 1 \leq 2^{[\sqrt{n}]}T([n/2]) + 2^{[\sqrt{n}]} + 1 \leq 2 \cdot 2^{2\sqrt{n}}(T([n/2]) + 1).
\]

Iterating this formula down to \( n = 2 \), we get that

\[
T(n) \leq 2 \cdot 2^{2\sqrt{n}} \cdot 2^{2\sqrt{n/2}} \cdots 2^2 \leq 2 \cdot 2^{2\sqrt{n(1+1/\sqrt{2}+1/\sqrt{\sqrt{2}}+\ldots)}} = 2^{O(\sqrt{n})}.
\]