1. Which of the following pairs of statements are logically equivalent?

(a) If it is raining then the ground is wet.
   If the ground is not wet then it isn’t raining.

(b) If it is raining then the ground is wet.
    It is not raining or the ground is wet.

(c) It is not true that it raining and the ground is wet.
    It is not raining or the ground is not wet.

(d) If you are rich then you are famous.
    You are not famous if you are not rich.

Solution:

(a) Let \( \text{rain} \) be the proposition “It is raining” and \( \text{wet} \) be the proposition “The ground is wet”.
   The two propositions are
   \[
   \text{rain} \rightarrow \text{wet} \quad \text{(NOT wet)} \rightarrow \text{(NOT rain)}.
   \]
   We can check that they are equivalent by writing out a truth table:
   \[
   \begin{array}{ccc}
   \text{rain} & \text{wet} & \text{rain} \rightarrow \text{wet} \quad \text{NOT wet} \rightarrow \text{NOT rain} \\
   T & T & T \\
   T & F & F \\
   F & T & T \\
   F & F & T \\
   \end{array}
   \]
   They are logically equivalent.

(b) Here, the two propositions are
   \[
   \text{rain} \rightarrow \text{wet} \quad \text{(NOT rain)} \text{ OR } \text{wet}
   \]
   and the truth table is
   \[
   \begin{array}{ccc}
   \text{rain} & \text{wet} & \text{rain} \rightarrow \text{wet} \quad \text{(NOT rain)} \text{ OR } \text{wet} \\
   T & T & T \\
   T & F & F \\
   F & T & T \\
   F & F & T \\
   \end{array}
   \]
   Again, the two are logically equivalent.

(c) The two propositions are
   \[
   \text{NOT (rain AND wet)} \quad \text{(NOT rain)} \text{ OR (NOT wet)}
   \]
<table>
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<tr>
<th>rain</th>
<th>wet</th>
<th>NOT (rain AND wet)</th>
<th>(NOT rain) OR (NOT wet)</th>
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The two statements have same value on each cases. So they are logically equivalent.

(d) Let rich be the proposition ”You are rich”, famous be the proposition “You are famous”. We need to decide the equivalence of

\[ rich \longrightarrow famous \quad (\text{NOT rich}) \longrightarrow (\text{NOT famous}) \]

<table>
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<tr>
<th>rich</th>
<th>famous</th>
<th>rich \rightarrow famous</th>
<th>NOT rich \rightarrow NOT famous</th>
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The two differ in the second and third row, so they are not logically equivalent.

2. “Jason, I heard that you are learning logic”, says Bob. “Please translate this strange note into English for me. You can use the this dictionary:”

- \( \text{SPY}(x) \) means “\( x \) is a spy.”
- \( \text{FRIEND}(x, y) \) means “\( x \) and \( y \) are friends.”
- \( \text{EXPOSE}(x, y) \) means “Operation \( x \) is exposed by \( y \).”
- \( \text{FAIL}(x) \) means “Operation \( x \) fails.”

The note says:

(a) \( \forall x: (x \neq \text{Alice}) \longrightarrow \text{SPY}(x) \)
(b) \( \forall x, y: (\text{SPY}(x) \text{ AND } \text{FRIEND}(x, y)) \longrightarrow \text{SPY}(y) \)
(c) \( \forall x \exists y: \text{EXPOSE}(x, y) \longrightarrow \text{FAIL}(x) \)
(d) \( \forall x \forall y: \text{EXPOSE}(x, y) \longrightarrow \text{SPY}(y) \)

Solution:

(a) For every person \( x \), if \( x \) is not Alice then \( x \) is a spy. In plain English, everyone except Alice is a spy.

(b) For every person \( x \) and every person \( y \), if \( x \) is a spy and \( x \) and \( y \) are friends, then \( y \) is also a spy. In plain English, a friend of a spy is always a spy.

(c) For every operation \( x \), if \( x \) is exposed by someone then \( x \) will fail. In plain English, every exposed operation fails.

(d) For every operation \( x \) and every person \( y \), if \( y \) exposes \( x \) then \( y \) is a spy. In plain English, anyone who exposes an operation is a spy.
3. Express the following statements about positive integers using quantifiers and the predicates $P(x)$ which stands for “$x$ is a prime number” and $E(x)$ which stands for “$x$ is an even number”.

(a) Every even integer greater than 2 is the sum of two primes.
(b) There is exactly one even prime number.
(c) If you raise 2 to the power of a prime number and subtract one, you always get a prime number.
(d) Two prime numbers are called twin primes if they differ by exactly two. There are infinitely many twin primes.

Solution:

(a) This proposition says that for every integer $n$ greater than 2 there exist two prime numbers $x$ and $y$ so that $n = x + y$. Symbolically, we write

$$\forall n: n > 2 \land E(n) \rightarrow \exists x, y: P(x) \land P(y) \land n = x + y.$$  
(b) “There exists exactly one” means “There exists at least one” and “every two are equal”. We can write this as

$$\exists n: P(n) \land E(n) \land (\forall m, n: (P(m) \land E(m) \land P(n) \land E(n)) \rightarrow m = n).$$

Another solution is to say “There exists an even prime number $n$ and every even prime number $m$ is equal to this $n$”. We can write this as

$$\exists n: P(n) \land E(n) \land (\forall m: (P(m) \land E(m)) \rightarrow m = n).$$

(c) This proposition says if $n$ is prime, then $2^n - 1$ is also prime:

$$\forall n: P(n) \rightarrow P(2^n - 1).$$

(d) “There are infinitely many twin primes” means that given any number $n$, no matter how large, there are twin primes $m$ and $m + 2$ greater than it:

$$\forall n \exists m: m > n \land P(m) \land P(m + 2).$$

4. This is about the adventures of Alex, Bob, and Chris, three students of logic:

(a) Alex, Bob and Chris go to a village to collect rotten tomatoes and they get lost. They ask a villager for help. “Don’t worry”, the villager says, “I will help you if and only if I tell the truth.” Will the villager help them or not?

(b) Alex, Bob, and Chris go to a cafe. The waiter asks, “Does everyone want an espresso?” Alex says, “I am not sure if everyone wants an espresso.” Then Bob says, “I am not sure if everyone wants an espresso.” Finally, Chris says, “Yes, please bring everyone an espresso.” Can you explain what happened?
(c) **(Extra credit)** Alex, Bob, and Chris stand in line, with Chris in front. Jason takes three white hats and two black hats, puts a hat on each head, and tosses out the remaining two hats. Alex observes Bob’s and Chris’s hats and announces “I don’t know the colour of my hat.” Bob sees Chris’s hat and says “I don’t know the colour of my hat.” What is the colour of Chris’s hat?

**Solution:**

(a) Let $p$ be the proposition “the villager tells the truth” and $q$ be the proposition “the villager will help Alex, Bob, and Chris.” Let’s look at the truth table for $p$ IFF $q$:

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If the villager tells the truth, then $p$ and $p$ IFF $q$ are both true. Looking at the truth table, we must be in the first row and we see that $q$ must also be true.

If the villager doesn’t tell the truth, then $p$ and $p$ IFF $q$ are both false. Looking at the truth table, we must be in the third row and $q$ must also be true.

In either case, $q$ is true, so Alex, Bob, and Chris do not need to worry.

(b) Let $a$, $b$, $c$ be the propositions “Alex wants an espresso”, “Bob wants an espresso”, and “Chris wants an espresso”, respectively. The waiter is asking “Is $a$ AND $b$ AND $c$ true?”

If $a$ were false, then $a$ AND $b$ AND $c$ would also be false, so Alex would have said “no”. Since he didn’t say “no”, $a$ must be true. (He can’t say “yes” because he doesn’t know the truth of $b$ and $c$).

Now, assuming $a$ is true, if $b$ were false, $a$ AND $b$ AND $c$ would also be false, so Bob would have said “no”. Since he didn’t say “no”, $b$ must also be true.

Therefore, when Chris replies, he knows that $a$ and $b$ must both be true. He also wants an espresso so he says yes.

(c) Since Alex says he doesn’t know he has the colour of his hat, it can’t be that both Bob and Chris had black hats, for then Alex would know he had a white hat. Therefore, after Alex speaks, Bob and Chris know that don’t both have black hats.

If Chris had a black hat, then Bob would have known he has a white hat at the time he speaks. Since he says he doesn’t know the colour of his hat, Chris must have a white hat.

5. Jason is asking his friends whether they will join Occupy Central or not. He has collected the following information from four of his friends, Alex, Bob, Chris and David.

- Alex and Bob will not join together.
- If Chris joins then Bob will join.
- If David doesn’t join then Chris will join.
- If David joins then Bob will join.
He wants to know who will join and he asks you to help him. Show your deduction clearly and carefully.

**Solution:** Let $A$, $B$, $C$, $D$ be the propositions that Alex, Bob, Chris, and David will join Occupy Central, respectively. We know that

\[
\begin{align*}
A & \rightarrow \neg B \\
B & \rightarrow \neg A \\
C & \rightarrow B \\
\neg B & \rightarrow \neg C \\
\neg D & \rightarrow C \\
\neg C & \rightarrow D \\
D & \rightarrow B \\
\neg B & \rightarrow \neg D
\end{align*}
\]

Propositions (4), (6), and (7) tell us

\[
\neg B \rightarrow \neg C \rightarrow D \rightarrow B
\]

so $\neg B$ implies $B$. Since false implies true but true does not imply false, $\neg B$ must be false and $B$ must be true. Then from (2), we find out that $A$ is false.

We conclude that $A$ is false and $B$ is true. What about $C$ and $D$? Let’s try all possible truth assignments to $C$ and $D$ and see which ones are consistent with propositions (1)-(8):

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We see that there are two possibilities corresponding to the second and third row. So either Bob and Chris will go, or Bob and David will go.
6. Jason is organising the distribution of pizza lunches for Occupy Central. Here is the network of roads that can be used to transport food:

An arrow pointing from $X$ to $Y$ with label $t$ indicates that at most $t$ pizza trucks can pass from $X$ to $Y$ before the officials figure out something suspicious is going on. For example, Jason can route 3 but not 4 trucks from Pizza Hut through B and then D to Central.

(a) How should Jason route the pizza trucks in order to maximise the number of them that reach Central (without alerting the authorities)?

(b) Explain convincingly why it is not possible for Jason to route any additional trucks.

(a) This is one possible scheme by which Jason routes 14 trucks:

(b) Let’s look at how many trucks can pass from Pizza Hut and B on one side and all the other points A, C, D, E, Central on the other side. The roads connecting the two sides are marked in black:
All the trucks from Pizza Hut to Central must pass through at least one of the roads marked in black. Since these roads have capacity for at most $9 + 1 + 1 + 3 = 14$ trucks, no more than 14 trucks can make it from Pizza Hut to Central.