5. Conditioning and Independence

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Conditional PMF

Let $X$ be a random variable and $A$ be an event.

The conditional PMF of $X$ given $A$ is

$$P(X = x \mid A) = \frac{P(X = x \text{ and } A)}{P(A)}$$
What is the PMF of a 6-sided die roll given that the outcome is even?

\[ P_X(x) = \frac{1}{6} \quad \text{for} \quad x = 1, 2, 3, 4, 5, 6 \]

\[ P(X=x \mid E) = \frac{1}{3} \quad \text{for} \quad x = 2, 4, \text{ or } 6 \]

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_X(x))</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
<td>(\frac{1}{6})</td>
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</tbody>
</table>

| x | 2 | 4 | 6 |
|---|---|---|
| \(P(X=x \mid E)\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) | \(\frac{1}{3}\) |
You flip 3 coins. What is the PMF number of heads given that there is at least one?

**UNCONDITIONAL PMF**

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
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<th>3</th>
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</thead>
<tbody>
<tr>
<td>$P(X=x)$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{1}{8}$</td>
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</table>

$P(X=x|A)$

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>$P(X=x</td>
<td>A)$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{3}{7}$</td>
<td>$\frac{1}{7}$</td>
</tr>
</tbody>
</table>

$x = 1, 2, \text{or } 3$: $P(X=x|A) = \frac{P(X=x \text{ and } A)}{P(A)} = \frac{P(X=x)}{P(X=1)+P(X=2)+P(X=3)}$
Conditioning on a random variable

Let $X$ and $Y$ be random variables.

The **conditional PMF** of $X$ given $Y$ is

$$ P(X = x \mid Y = y) = \frac{P(X = x \text{ and } Y = y)}{P(Y = y)} $$

$$ p_{X \mid Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)} $$

For fixed $y$, $p_{X \mid Y}$ is a PMF as a function of $x$. $\sum_x p_{X \mid Y}(x \mid y) = 1$ for all $y$. 
Roll two 4-sided dice. What is the PMF of the sum given the first roll?

\[ X = \text{FIRST ROLL} \]
\[ S = \text{SUM} \]

\[ P(S=s \mid X=x) = \frac{P(S=s, X=x)}{P(X=x)} \]

\[
\begin{array}{cccccccc}
1 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 \\
2 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\
3 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\
4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\
\end{array}
\]
Roll two 4-sided dice. What is the PMF of the sum given the first roll?

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
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<th>8</th>
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<tr>
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<tr>
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<td>0</td>
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<tr>
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<td>1/16</td>
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</table>

**Joint PMF**

**Marginal PMF**

<table>
<thead>
<tr>
<th>F</th>
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<th>1/4</th>
<th>1/4</th>
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</thead>
<tbody>
<tr>
<td>P(F=f)</td>
<td>1/4</td>
<td>1/4</td>
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</tbody>
</table>

**Conditional PMF**

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<td>1/4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
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<td></td>
</tr>
<tr>
<td>4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
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</tbody>
</table>
Roll two 4-sided dice. What is the PMF of the first roll given the sum?
The conditional expectation of $X$ given event $A$ is

$$E[X \mid A] = \sum_x x \cdot P(X = x \mid A)$$

The conditional expectation of $X$ given $Y = y$ is

$$E[X \mid Y = y] = \sum_x x \cdot P(X = x \mid Y = y)$$
You flip 3 coins. What is the expected number of heads given that there is at least one?

\[
\begin{array}{c|c|c|c}
   & 1 & 2 & 3 \\
\hline
P(X=x|A) & \frac{3}{7} & \frac{2}{7} & \frac{1}{7} \\
\end{array}
\]

\[
E[X|A] = 1 \cdot \frac{3}{7} + 2 \cdot \frac{2}{7} + 3 \cdot \frac{1}{7} = \frac{12}{7}
\]
Total Expectation Theorem

\[ E[X] = E[X \mid A] P(A) + E[X \mid A^c] P(A^c) \]

Proof:

\[
\begin{align*}
\sum_x x \cdot P(X=x \mid A) P(A) + \sum_x x \cdot P(X=x \mid A^c) P(A^c) \\
= \sum_x x \cdot (P(X=x \mid A) P(A) + P(X=x \mid A^c) P(A^c)) \\
= \sum_x x \cdot P(X=x) \\
= E[X]
\end{align*}
\]
Total Expectation Theorem (general form)

If \( A_1, \ldots, A_n \) partition \( \Omega \) then

\[
E[X] = E[X \mid A_1]P(A_1) + \ldots + E[X \mid A_n]P(A_n)
\]

In particular

\[
E[X] = \sum_y E[X \mid Y = y] P(Y = y)
\]
<table>
<thead>
<tr>
<th>type</th>
<th>average time on facebook</th>
<th>% of visitors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>30 min</td>
<td>60%</td>
</tr>
<tr>
<td></td>
<td>60 min</td>
<td>30%</td>
</tr>
<tr>
<td></td>
<td>10 min</td>
<td>10%</td>
</tr>
</tbody>
</table>

average visitor time = \( E[X_{A}]P(A) + E[X_{B}]P(B) + E[X_{C}]P(C) = 30 \cdot 60\% + 60 \cdot 30\% + 10 \cdot 10\% \)
You play 10 rounds of roulette. You start with $100 and bet 10% on red in every round.

On average, how much cash will remain?

\[ X_t = \text{CASH AFTER } t \text{ ROUNDS} \]

\[
E[X_t] = E[X_t|W_t]P(W_t) + E[X_t|W_t^c]P(W_t^c)
\]

\[
= 110 \cdot \frac{18}{37} + 90 \cdot \frac{19}{37}
\]

\[
E[X_t] = E[X_{t-1}]P(W_t) + E[X_{t-1}]P(W_t^c)
\]

\[
= E[1.1X_{t-1}] \frac{18}{37} + E[0.9X_{t-1}] \frac{19}{37}
\]

\[
= (1.1 \cdot \frac{18}{37} + 0.9 \cdot \frac{19}{37}) \cdot E[X_{t-1}]
\]

\[
\approx 0.997 E[X_{t-1}] \quad E[X_{10}] \approx 0.997^{10} \cdot 100 \approx 97.3
\]
You flip 3 coins. What is the expected number of heads given that there is at least one?

\[ X = \text{NUMBER OF HEADS} \]
\[ A = \text{AT LEAST ONE} \]

\[ E[X] = E[X|A]P(A) + E[X|A^c]P(A^c) \]
\[ \frac{3}{2} = E[X|A] \cdot \frac{7}{8} + 0 \cdot \frac{1}{8} \]

\[ E[X|A] = \frac{3/2}{7/8} = \frac{12}{7}. \]
Mean of the Geometric

\[ X = \text{Geometric}(\theta) \text{ random variable} \]

\[ E[X] = p \cdot 1 + (1-p) \cdot p \cdot 2 + \ldots \]

**Another Way:**

\[ A = \text{Trial 1 Succeeds} \]

\[ E[X] = E[X|A]P(A) + E[X|A^c]P(A^c) \]

\[ = 1 \cdot p + E[1+Y|A^c] \cdot (1-p) \]

\[ = p + (1+E[Y|A^c]) \cdot (1-p) \]

\[ = p + (1+E[X]) \cdot (1-p) \]

\[ E[X] = \frac{1}{\theta} \]
Variance of the Geometric

\[ X = \text{Geometric}(\theta) \text{ random variable} \]

\[
\text{Var}[X] = E[(X - \frac{1}{\theta})^2]
= E[(X - \frac{1}{\theta})^2 | A] \cdot p + E[(X - \frac{1}{\theta})^2 | A^c] \cdot (1-p)
= (1 - \frac{1}{\theta})^2 \cdot p + E[(1 + \frac{1}{\theta} - \frac{1}{\theta})^2 | A^c] \cdot (1-p)
= (1 - \frac{1}{\theta})^2 \cdot p + E[1 + 2(Y - \frac{1}{\theta}) + (Y - \frac{1}{\theta})^2 | A^c] \cdot (1-p)
= (1 - \frac{1}{\theta})^2 \cdot p + (1 + \text{Var}[X]) \cdot (1-p)
\]

\[
\text{Var}[X] = \left(\frac{1}{\theta} - 1\right)^2 + \left(\frac{1}{\theta} - 1\right) = \frac{1-p}{\theta^2}
\]
stay or switch?
Bob should **stay** because...

DON'T LOSE WHAT YOU HAVE

DOESN'T MAKE A DIFFERENCE

Bob should **switch** because...

\[
E[Y] = \frac{1}{2} E[2X] + \frac{1}{2} E[X/2]
\]

\[
= \frac{1}{2} \cdot 2E[X] + \frac{1}{2} \cdot \frac{1}{2} E[X]
\]

\[
= \frac{5}{4} E[X]
\]
A person decides between two envelopes:

- $10
- $20

There's a 50% chance of each value. The expected values are:

- $X$: $\frac{1}{2} \times 0 + \frac{1}{2} \times 20 = 10$
- $Y$: $\frac{1}{2} \times 20 + \frac{1}{2} \times 10 = 15$

The person should always choose the envelope with $20$.
### Joint PMF

<table>
<thead>
<tr>
<th>$X$</th>
<th>$Y$</th>
<th>$P(X=x, Y=y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>20</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>20</td>
<td>40</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>40</td>
<td>20</td>
<td>$\frac{1}{4}$</td>
</tr>
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</table>

### Expected Value

$E[Y|X=20] = \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 40 = 25 \quad \text{SWITCH!}$
Independent random variables

Let $X$ and $Y$ be **discrete** random variables.

$X$ and $Y$ are **independent** if

$$P(X = x, Y = y) = P(X = x) P(Y = y)$$

for all possible values of $x$ and $y$.

In PMF notation, $p_{XY}(x, y) = p_X(x) p_Y(y)$ for all $x, y$. 

Independent random variables

$X$ and $Y$ are independent if

$$P(X = x \mid Y = y) = P(X = x)$$

for all $x$ and $y$ such that $P(Y = y) > 0$.

In PMF notation, $p_{X \mid Y}(x \mid y) = p_X(x)$ if $p_Y(y) > 0$. 
Alice tosses 3 coins and so does Bob. Alice gets $1 per head and Bob gets $1 per tail.

Are their earnings independent?

YES

\[ P(B = b | A = a) = P(B = b) = \binom{3}{b} \cdot \frac{1}{8}. \]
Now they toss the same coin 3 times. Are their earnings independent?

\[ A + B = 3 \]

\[ P(B = 3 \mid A = 5) = 0 \]

\[ P(B = 3) = \frac{1}{8} \]

\text{DEPENDENT}
Expectation and independence

$X$ and $Y$ are independent if and only if

$$
E[f(X)g(Y)] = E[f(X)] \cdot E[g(Y)]
$$

for all real valued functions $f$ and $g$. 
Expectation and independence

In particular, if $X$ and $Y$ are independent then

$$E[XY] = E[X] E[Y]$$

Not true in general!
Variance of a sum

Recall $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$\text{Var}[X + Y] = \mathbb{E}[(X + Y)^2] - \mathbb{E}[X+Y]^2$$

\[
\begin{aligned}
\mathbb{E}[X^2 + 2XY + Y^2] &= \mathbb{E}[X^2] + \mathbb{E}[2XY] + \mathbb{E}[Y^2] \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Y^2] \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}
\]

\[
\begin{aligned}
(\mathbb{E}[X] + \mathbb{E}[Y])^2 &= \mathbb{E}[X]^2 + \mathbb{E}[Y]^2 + 2\mathbb{E}[X]\mathbb{E}[Y] \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] + 2\mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}
\]

$$\text{Var}[X] + \text{Var}[Y]$$
Variance of a sum

\[ \text{Var}[X_1 + \ldots + X_n] = \text{Var}[X_1] + \ldots + \text{Var}[X_n] \]

if every pair \( X_i, X_j \) is independent.

Not true in general!
Variance of the Binomial \( \text{Binomial}(n, p) \)

\[
X = X_1 + X_2 + \ldots + X_n
\]

\[\uparrow \quad \text{1 IF TRIAL SUCCEEDS} \]
\[\text{0 IF TRIAL FAILS} \]

\[
\text{Var}[X] = \text{Var}[X_1] + \text{Var}[X_2] + \ldots + \text{Var}[X_n]
\]

\[
\text{Var}[X_i] = E[X_i^2] - E[X_i]^2 = p - p^2 = p(1-p)
\]

\[
\text{Var}[X] = n \cdot p \cdot (1-p)
\]
Sample mean

$X = \# \text{ POLLED PEOPLE THAT LIKE}$

$\text{Binomial}(n, p)$

$E[X] = np$

$\sigma = \sqrt{np(1-p)}$

Ex. $p = 50\%$

$E[X] = \frac{n}{2}$

$\sigma = \frac{\sqrt{n}}{2}$

$n = 100$

$50$

$n = 10000$

$5000$

$5$

$50$
\[ \rho = 0.35 \]
Variance of the Poisson

**Poisson(λ) approximates Binomial(n, λ/n) for large n**

\[ p(k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k = 0, 1, 2, 3, \ldots \]

\[
\text{Var} \left[ \text{Binomial}(n, p) \right] = n \cdot p \cdot (1-p)
\]

\[
\lim_{n \to \infty} n \cdot p \cdot (1-p) = \lim_{n \to \infty} n \cdot \frac{1}{n} \cdot (1-\frac{1}{n}) = \lambda \quad (p = \frac{1}{n})
\]

\[
\text{Var} \left[ \text{Poisson}(\lambda) \right] = \lambda \quad \sigma = \sqrt{\lambda}
\]
Independence of multiple random variables

\(X, Y, Z\) independent if

\[P(X = x, Y = y, Z = z) = P(X = x) P(Y = y) P(Z = z)\]

for all possible values of \(x, y, z\).

\(X, Y, Z\) independent if and only if

\[E[f(X)g(Y)h(Z)] = E[f(X)] E[g(Y)] E[h(Z)]\]

for all \(f, g, h\).

Usual warnings apply.