Practice questions

1. A point is chosen uniformly at random inside a triangle with base 1 and height 1. Let \( X \) be the distance from the point to the base of the triangle. Find the CDF and the PDF of \( X \). *(Textbook problem 3.2.5)*

**Solution:** The PDF of the point is uniform over the triangle which has area \( \frac{1}{2} \), so it has value 2 inside the triangle and zero outside. The event \( X > x \) consists of all the points in the triangle that are at distance more than \( x \) from the base, which is itself a triangle of base and height \( 1 - x \). Therefore \( P(X > x) = 2(1 - x)^2/2 = (1 - x)^2 \). The CDF is \( P(X \leq x) = 2x - x^2 \) and the PDF is \( f_X(x) = \frac{d}{dx} P(X \leq x) = 2(1 - x) \).

2. There are 100 students in class. The arrival times of students (in minutes) are exponential random variables with rate \( \lambda = 0.2 \), starting from 09:20.

   (a) What is the expected number of students that have arrived by 09:30?
   
   (b) Assuming students’ arrivals are independent, what is the probability that everyone has made it by 09:45?

**Solution:** Let \( T_i \) be the arrival time of student \( i \). The CDF of \( T_i \) is \( F_{T_i}(t) = 1 - e^{-\lambda t} \).

   (a) The probability that a given student has arrived by 09:30 is \( P(T_i \leq 10) \approx 0.865 \). The number of students that have arrived by 09:30 is \( X_1 + \cdots + X_{100} \) where \( X_i \) is an indicator random variable for the event \( T_i \leq 10 \). By linearity of expectation the expected number of such students is the sum of \( P(T_i \leq 10) \) as \( i \) ranges over the 100 students, which is about 86.5.

   (b) The probability that any given student has arrived by 09:45 is \( p = P(T_i \leq 25) \approx 0.993 \). The number of students arriving before 09:45 is a Binomial(100, \( p \)) random variables, so the probability they all arrived by this time is \( (1 - p)^{100} \approx 0.509 \).

3. Three points are dropped at random on the perimeter of a circle with 1 unit circumference.

   (a) What is the probability that they all fall within 1/4 of a unit of one another?
   
   (b) What is the probability that every pair of them is at least 1/4 of a unit apart?
   
   **(Hint:** Fix one of the three points.)

**Solution:** Let’s call the three points \( a \), \( b \), and \( c \). By symmetry, we can position \( a \) on the circle in an arbitrary way. Let \( X \) and \( Y \) be the positions of \( b \) and \( c \) relative to \( a \) clockwise along the circle. We model \( X \) and \( Y \) as independent Uniform(0, 1) random variables.

   (a) The event \( E \) is the intersection of events \( A \), \( B \), \( C \) described by the predicates: (1) \( x \in [0, 1/4] \cup [3/4, 1] \) (\( b \) is close to \( a \)); (2) \( y \in [0, 1/4] \cup [3/4, 1] \) (\( c \) is close to \( a \)); and (3) \( |x - y| \in [0, 1/4] \cup [3/4, 1] \) (\( b \) is close to \( c \), clockwise or counterclockwise). \( A \cap B \cap C \) is the shaded set in the following diagram and has probability 3/16.
(b) The event $E'$ of interest is now $A' \cap B' \cap C'$, where $A', B', C'$ are the sets (1) $x \in [1/4, 3/4]$ ($b$ is far from $a$); (2) $y \in [1/4, 3/4]$ ($c$ is far from $a$); and (3) $|x - y| \in [1/4, 3/4]$ ($b$ is far from $c$). This is represented by the shaded region below and has area $1/16$.

Another way to solve part (b) (or to check your answer) is via the axioms of probability. The complement of $E'$ equals $A \cup B \cup C$ (some pair of points is close), so by inclusion-exclusion:

$$P(E'^c) = P(A \cup B \cup C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Here, $A$ is the event that points $a$ and $b$ are less than $1/4$ of an inch apart, so $P(A) = 1/2$. For the same reason $P(B) = P(C) = 1/2$. The events $A, B$ are independent so $P(A \cap B) = P(A)P(B) = 1/4$. For the same reason $P(B \cap C) = P(C \cap A) = 1/4$. In part (a) we calculated that $P(A \cap B \cap C) = 3/16$, so

$$P(E'^c) = 3 \times \frac{1}{2} - 3 \times \frac{1}{4} + \frac{3}{16} = \frac{15}{16},$$

and $P(E') = 1/16$.

4. A coin has probability $P$ of being heads, where $P$ itself is a Uniform$(0, 1)$ random variable. Find the PMF of the number of heads after performing two independent coin flips.

**Solution:** Let $N$ be the number of heads in two coin flips. The conditional PMF of $X$ given $P$ is $f_{X|P}(x|p) = \binom{2}{x}p^x(1-p)^{2-x}$. As $P$ is a Uniform$(0, 1)$ random variable, its PDF is $f_P(p) = 1$ when $0 \leq p \leq 1$ and 0 otherwise. By the total probability theorem, the (unconditional) PMF of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|P}(x|p)f_P(p)dp = \int_0^1 \binom{2}{x}p^x(1-p)^{2-x} dp.$$
It remains to calculate this integral for $x = 0, 1, 2$:

\[
\begin{align*}
    f_X(0) &= \int_{0}^{1} (1 - p)^2 dp = \frac{1}{3} \\
    f_X(1) &= \int_{0}^{1} 2p(1 - p) dp = \frac{1}{3} \\
    f_X(2) &= \int_{0}^{1} p^2 dp = \frac{1}{3}.
\end{align*}
\]

5. Here is a way to solve Buffon’s needle problem without calculus. Recall that an $\ell$ inch needle is dropped at random onto a lined sheet, where the lines are one inch apart.

(a) Let $A$ be the number of lines that the needle hits. Let $B$ be the number of times that a polygon of perimeter $\ell$ hits a line. Show that $E[A] = E[B]$. (Hint: Use linearity of expectation.)

(b) Assume that $\ell < \pi$. Calculate the expected number of times that a circle of perimeter $\ell$ hits a line.

(c) Assume that $\ell < 1$. Use part (a) and (b) to derive a formula for the probability that the needle hits a line. (Hint: The number of hits is a Bernoulli random variable.)

**Solution:**

(a) Suppose the polygon has $n$ edges of length $a_1, a_2, \ldots, a_n$. Break up the needle into segments of lengths $a_1, a_2, \ldots, a_n$. Let $A_i$ and $B_i$ be the number of lines hit by the $i$-th segment of the needle and the $i$-th edge of the polygon, respectively. Then

\[
    A = A_1 + \cdots + A_n \quad \text{and} \quad B = B_1 + \cdots + B_n.
\]

By linearity of expectation

\[
    E[A] = E[A_1] + \cdots + E[A_n] \quad \text{and} \quad E[B] = E[B_1] + \cdots + E[B_n].
\]

Since the $i$-th edge of the polygon and the $i$-th segment of the needle are identical, $E[A_i] = E[B_i]$. It follows that $E[A] = E[B]$.

(b) Let $C$ be the number of times a circle intersects a line. We calculate the p.m.f. of $C$. Let $d$ be the line segment representing the diameter of the circle that is perpendicular to the lines on the sheet. Since $\ell < \pi$, the length of $d$ is less than 1. The circle hits a line twice if $d$ crosses a line, once if $d$ touches one of the lines, and zero times if $d$ does not intersect any of the lines. The probability that $d$ crosses a line is exactly the length of $d$, namely $\ell/\pi$, and the probability that $d$ touches a line is zero. Summarizing, the p.m.f. of $C$ is

\[
    \begin{array}{c|c|c|c}
        c & 0 & 1 & 2 \\
        \hline
        P(C = c) & 1 - \ell/\pi & 0 & \ell/\pi
    \end{array}
\]

Therefore $E[C] = 2\ell/\pi$.

(c) If we view the circle as a polygon with infinitely many sides, putting together part (a) and (b) we get that $E[A] = E[C] = 2\ell/\pi$. Since $\ell < 1$, the number of times the needle intersects a line is a 0/1 valued random variable, so $E[A] = P(A = 1) = P($the needle hits a line$)$. Therefore the probability the needle hits a line is exactly $2\ell/\pi$. 