Practice Final 1

1. The joint probability density function of the lifetimes $X$ and $Y$ of two connected components in a machine is

$$f_{X,Y}(x, y) = \begin{cases} xe^{-x(1+y)}, & x \geq 0, y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

(a) What is the probability that the lifetime $X$ of the first component exceeds 3?

Solution: $P(X > 3) = \int_3^\infty \int_0^\infty xe^{-x(1+y)}dydx = \int_3^\infty e^{-x}dx = e^{-3} = 0.05$.

(b) Are $X$ and $Y$ independent? Justify your answer.

Solution: No. If $X$ and $Y$ were independent then $f_{X,Y}(1, 1)f_{X,Y}(2, 2) = 2e^{-8}$ and $f_{X,Y}(1, 2)f_{X,Y}(2, 1) = 2e^{-7}$ should both be equal to the same value $f_X(1)f_X(2)f_Y(1)f_Y(2)$. Alternatively one can calculate the marginal PDFs

$$f_X(x) = \int_0^\infty xe^{-x(1+y)}dy = e^{-x},$$

$$f_Y(y) = \int_0^\infty xe^{-x(1+y)}dx = \frac{1}{(1+y)^2}$$

for $x, y \geq 0$ and conclude that $f_X(x)f_Y(y)$ is not the same function as $f_{X,Y}(x, y)$.

2. A radio station gives a gift to the third caller who knows the birthday of the radio talk show host. Each caller has a 0.7 probability of guessing the host’s birthday, independently of other callers.

(a) What is the probability mass function of the number of calls necessary to find the winner?

Solution: $n$ calls are necessary to find the winner if the $n$-th guess is correct and there are exactly two correct guesses among the first $n - 1$. The probability of this is

$$P(N = n) = \binom{n-1}{2} \cdot 0.7^3 \cdot 0.3^{n-3}.$$

(b) What is the probability that the station will need five or more calls to find a winner?

Solution: No winner has been found in the first four calls if the number of correct guesses in those calls is 0, 1, or 2. The probability of this is

$$P(N \geq 5) = 0.3^4 + 4 \cdot 0.7 \cdot 0.3^3 + \binom{4}{2} \cdot 0.7^2 \cdot 0.3^2 = 0.3483.$$
2. Alice sends a message $a$ that equals $-1$ or $1$. Bob receives the value $B$ which is a Normal random variable with mean $a$ and standard deviation $0.5$. Bob guesses that Alice sent $1$ if $B > 0.5$, that Alice sent $-1$ if $B < -0.5$, and declares failure otherwise (when $|B| \leq 0.5$).

(a) What is the probability that Bob declares failure?

**Solution:** For either message, failure occurs when a normal random variable is between $1$ and $3$ standard deviations from the mean on one side. If $N$ is a Normal($0, 1$) random variable then

$$P(1 \leq N < 3) = P(N < 3) - P(N < 1) \approx 0.9987 - 0.8413 = 0.1574.$$ 

(b) Given that Bob didn’t declare failure, what is the probability that his guess is correct?

**Solution:** By symmetry we may assume Alice sent $1$. The event “Bob’s guess is correct and failure didn’t occur” happens when $B$ takes value $0.5$ or larger, or when a Normal($0, 1$) random variable $N$ takes value at most $1$, which is approximately $0.8413$. Therefore the conditional probability that Bob’s guess is correct is about $0.8413/(1 - 0.1574) \approx 0.9985$.

4. The number of people who enter an elevator on the ground floor is a Poisson random variable with mean $10$. There are $20$ floors above (not including) the ground floor and each person is equally likely to get off on any one of them, independently of all others.

(a) What is the probability $p$ that the elevator doesn’t stop on the seventh floor?

**Solution:** The number of passengers bound for the seventh floor is a Poisson random variable with mean $10/20 = 1/2$, so the probability that no passengers land there is the probability that this random variable takes value zero, which is $p = e^{-1/2} \approx 0.6065$.

Alternatively, by conditioning on the number $N$ of passengers who enter the elevator and applying the total probability theorem,

$$p = \sum_{n=0}^{\infty} \frac{(19/20)^n}{n!} \cdot P(N = n) = \sum_{n=0}^{\infty} \frac{(19/20)^n}{n!} \cdot e^{-10} \cdot \frac{10^n}{n!} = e^{-10} \cdot \sum_{n=0}^{\infty} \frac{(10 \cdot 19/20)^n}{n!} = e^{-10} \cdot e^{10 \cdot 19/20} = e^{-1/2}.$$ 

(b) What is the expected number of stops that the elevator will make? (Express the answer in terms of $p$ in case you didn’t complete part (a).)

**Solution:** By linearity of expectation, the expected number of stops is the sum of the probabilities that the elevator stops on floor $1$ up to floor $20$. By part (a) each of these probabilities is $1 - e^{-1/2}$ so the answer is $20 \cdot (1 - e^{-1/2}) \approx 7.869$.

5. 500 balls are drawn without replacement from a bin with 600 black balls and 400 white balls.

(a) What is the expected number of black balls drawn?

**Solution:** In any given draw the probability that the ball is black is $600/1000 = 3/5$. By linearity of expectation the expected number of black balls drawn is $(3/5) \cdot 500 = 300$. 

(b) What is the variance of the number of black balls drawn?

**Solution:** The variance of the number of black balls is the sum of the 500 variances $v$ indicating that any given ball is black plus the 500 · 499 covariances $c$ indicating that any two given balls are black. Each individual variance is $v = 3/5 \cdot 2/5 = 6/25$, while each of the covariances equals
$$c = \frac{600}{1000} \cdot \frac{599}{999} - \left( \frac{600}{1000} \right)^2 \approx -2.402 \cdot 10^{-4},$$
from where the variance is $500v + 500 \cdot 499 \cdot c \approx 60.06$.

(c) Is the probability you drew fewer than 200 black balls more than 2%? Justify your answer.

**Solution:** The standard deviation of the number of black balls drawn is about $\sqrt{60.06} \approx 7.750$. In order to draw fewer than 200 black balls, the number of black balls drawn must be $100/7.750 \approx 12.90$ standard deviations lower than the mean. By Chebyshev’s inequality, the probability of this is at most $1/(12.90^2) \approx 0.006$. This is much less than 2%.

**Practice Final 2**

1. Urn A has 4 blue balls. Urn B has 1 blue ball and 3 red balls.

(a) You draw a ball from a random urn and it is blue. What is the probability that it came from urn A?

**Solution:** Let $B_1$ be the event the ball is blue and $A$ be the event the ball came from urn A. By Bayes’ rule

$$P(A|B_1) = \frac{P(B_1|A)P(A)}{P(B_1|A)P(A) + P(B_1|A^c)P(A^c)} = \frac{1 \cdot (1/2)}{1 \cdot (1/2) + (1/4) \cdot (1/2)} = \frac{4}{5}.$$

(b) You draw another ball from the same urn. What is the probability that the second ball is also blue?

**Solution:** Let $B_2$ be the event that the second ball is blue. By the total probability theorem and Bayes’ rule

$$P(B_2|B_1) = \frac{P(B_2 \cap B_1)}{P(B_1)} = \frac{P(B_2 \cap B_1|A)P(A) + P(B_2 \cap B_1|A^c)P(A^c)}{P(B_1|A)P(A) + P(B_1|A^c)P(A^c)}$$

$$= \frac{1 \cdot (1/2) + (1/4)^2 \cdot (1/2)}{1 \cdot (1/2) + (1/4) \cdot (1/2)} = \frac{17}{20}.$$

2. Computers A and B are linked through routers $R_1$ to $R_4$ as in the picture. Each router fails independently with probability 10%.

![Diagram of network]
(a) What is the probability there is a connection between $A$ and $B$?

**Solution:** Let $R_i$ be the event that router $i$ is operational. The event “there is a connection between $A$ and $B$” is $(R_1 \cup R_2) \cap (R_3 \cup R_4)$. By independence

$$P((R_1 \cup R_2) \cap (R_3 \cup R_4)) = P(R_1 \cup R_2) P(R_3 \cup R_4)$$

$$= (1 - P(R_1^c \cap R_2^c))(1 - P(R_3^c \cap R_4^c))$$

$$= (1 - 0.1^2)^2$$

$$= 0.9801.$$  

(b) Are the events “there is a connection between $A$ and $B$” and “exactly two routers fail” independent? Justify your answer.

**Solution:** No. The probability that there is a connection between $A$ and $B$ given that exactly two routers fail is $2/3$: Given that exactly two routers fail, the failed routers are equally likely to be any of the 6 pairs $R_1R_3, R_1R_4, R_2R_3, R_2R_4, R_1R_2, R_3R_4$, and there is a connection between $A$ and $B$ in the first 4 out of these 6 possibilities. This probability is not equal to the unconditional probability from part (a) and so the two events are not independent.

3. A bus takes you from $A$ to $B$ in 10 minutes. On average a bus comes once every 5 minutes. A taxi takes you in 5 minutes, and on average a taxi comes once every 10 minutes. Their arrival times are independent exponential random variables. A bus comes first.

(a) If you want to minimize the (expected) travel time, should you take this bus?

**Solution:** Yes. If you waited for a taxi your expected travel time would be the expected waiting time for the next taxi which is 10 minutes plus its travel time which is another 5 minutes for a total of 15 minutes.

(b) If you do take the bus, what is the probability that you made the wrong decision?

**Solution:** The probability of a wrong decision is the probability that a taxi arrives within the next five minutes, which is the probability that an Exponential($1/10$) random variable is less than 5, which is $1 - e^{-5/10} = 1 - e^{-1/2} \approx 39.35\%$.

4. 10 people toss their hats and each person randomly picks one. The experiment is repeated one more time.

(a) What is the probability that Bob picked his own hat both times?

**Solution:** By independence, the probability that Bob picked his hat both times is the product of the probabilities that he picked it in each trial, so it is $(1/10) \cdot (1/10) = 1/100.$

(b) Let $A$ be the event that at least one person picked their own hat both times. True or false: $P(A) > 25\%$? Justify your answer.

**Solution:** False. Let $X_i$ take value 1 if person $i$ picked their hat both times. $A$ occurs if $X = X_1 + \cdots + X_{10} \geq 1$. By part (a) and linearity of expectation, $E[X] = 10 \cdot (1/100) = 0.1$. By Markov’s inequality, $P[X \geq 1] \leq E[X]/1 = 0.1$ which is less than 25\%.
5. $X$ is a Normal\( (0, \Theta) \) random variable, where the prior PMF of the parameter $\Theta$ is $P(\Theta = 1/2) = 1/2$, $P(\Theta = 1) = 1/2$. You observe the following three independent samples of $X$: 1.0, 1.0, -1.0.

(a) What is the posterior PMF of $\Theta$?

**Solution:** By Bayes’ rule

$$f_{\Theta|X_1, X_2, X_3}(\theta|1.0, 1.0, -1.0) \propto f_{X_1, X_2, X_3}(1.0, 1.0, -1.0|\theta) P(\Theta = \theta) \propto \frac{1}{\theta^3} e^{-3/2\theta^2} P(\Theta = \theta).$$

As $\Theta$ is equally likely to take values 1/2 and 1, the posterior PMF is

$$f_{\Theta|X_1, X_2, X_3}(1/2|1.0, 1.0, -1.0) = \frac{8e^{-6}}{8e^{-6} + e^{-3/2}} f_{\Theta|X_1, X_2, X_3}(1|1.0, 1.0, -1.0) = \frac{e^{-3/2}}{8e^{-6} + e^{-3/2}}.$$

(b) What is the MAP estimate of $\Theta$?

**Solution:** As $e^{-3/2} \approx 0.2231$ is larger than $8e^{-6} \approx 0.0198$ the MAP estimate is $\hat{\Theta} = 1$.

(c) What is the posterior probability that $|X| \geq 1$?

**Solution:** The posterior probabilities of $\Theta$ are 1/2 with probability about 0.0198/(0.2231 + 0.0198) \approx 0.0815 and 1 with probability about 0.2231/(0.2231 + 0.0198) \approx 0.9185. By the total probability theorem the posterior probability of $|X| \geq 1$ is about

$$0.0815 \cdot P(|\text{Normal}(0, 1/2)| \geq 1) + 0.9185 P(|\text{Normal}(0, 1)| \geq 1)$$

$$\approx 0.0815 \cdot 2 P(\text{Normal}(0, 1) \geq 2) + 0.9185 \cdot 2 P(\text{Normal}(0, 1) \geq 1)$$

$$\approx 0.0815 \cdot 2 \cdot 0.023 + 0.9185 \cdot 2 \cdot 0.159$$

$$\approx 0.2958.$$

### Practice Final 3

1. Let $X, Y, Z$ be independent Binomial\( (2, \frac{1}{2}) \) random variables.

(a) What is the conditional PMF of $X$ conditioned on $X \neq Z$?

**Solution:** The joint PMF is

\[
P(X = 0, X \neq Z) = P(X = 0, Z = 1) + P(X = 0, Z = 2) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{16}.
\]

\[
P(X = 1, X \neq Z) = P(X = 1, Z = 0) + P(X = 1, Z = 2) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{4}{16}.
\]

\[
P(X = 2, X \neq Z) = P(X = 2, Z = 0) + P(X = 2, Z = 1) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{16}.
\]

The conditional PMF is the joint PMF normalized by $P(X \neq Z)$, which is

\[
P(X = 0|X \neq Z) = \frac{3}{10}, \quad P(X = 1|X \neq Z) = \frac{4}{10}, \quad P(X = 2|X \neq Z) = \frac{3}{10}.
\]
(b) Are \( X \) and \( Y \) independent conditioned on \( (X \neq Z) \) and \( (Y \neq Z) \)?

**Solution:** No. We show that

\[
P(X = 1 \mid X \neq Z, Y \neq Z) P(Y = 1 \mid X \neq Z, Y \neq Z) \neq P(X = 1, Y = 1 \mid X \neq Z, Y \neq Z).
\]

By symmetry the two probabilities on the left are the same. We calculate these expressions:

\[
P(X \neq Z, Y \neq Z) = \sum_{z \in \{0,1,2\}} P(X \neq Z, Y \neq Z \mid Z = z) P(Z = z)
\]

\[
= \left(\frac{3}{4}\right)^2 \cdot (1/4) + (1/2)^2 \cdot (1/2) + (3/4)^2 \cdot (1/4)
\]

\[
= 13/32,
\]

\[
P(X = 1, X \neq Z, Y \neq Z) = P(X = 1, Z \neq 1, Y \neq Z)
\]

\[
= P(X = 1) P(Z \neq 1, Y \neq Z)
\]

\[
= (1/2) \cdot (P(Y \neq 0, Z = 0) + P(Y \neq 2, Z = 2))
\]

\[
= (1/2) \cdot 2 \cdot (1/4) \cdot (3/4)
\]

\[
= 3/16,
\]

\[
P(X = 1, Y = 1, X \neq Z, Y \neq Z) = P(X = 1, Y = 1, Z \neq 1)
\]

\[
= (1/2)(1/2)(1/2)
\]

\[
= 1/8.
\]

By the conditional probability formula the expression on the left is \(((3/16)/(13/32))^2 \approx 0.2130\) and the one on the right is \((1/8)/(13/32) \approx 0.3077\). These are not equal.

2. Alice and Bob decide to meet somewhere. Alice’s arrival time \( A \) is uniform between 12:00 and 12:45. Bob’s arrival time \( B \) is uniform between 12:15 and 1:00. Their arrival times are independent.

(a) Let \( f_{A-B} \) be the PDF of \( A - B \). What is \( f_{A-B}(0) \)?

**Solution:** We model \( A \) and \( B \) as Uniform(0, 3/4) and Uniform(1/4, 1) random variables respectively (at the hour scale). By the convolution formula, \( f_{A-B}(0) = \int_{-\infty}^{\infty} f_A(t) f_B(t) \, dt \), where \( f_A, f_B \) are the PDFs of \( A \) and \( B \). \( f_A(t)f_B(t) \) takes value \((4/3)^2\) when \( t \) is between 1/4 and 3/4 and 0 otherwise, so the integral equals \((1/2) \cdot (4/3)^2 = 8/9\). (If time is scaled in minutes the answer is 60 times smaller.)

(b) What is the probability that Bob arrives before Alice?

**Solution:** The event that Bob arrives before Alice is the value of the integral \( \int_{a\geq b} f_A(a) f_B(b) \, dadb \). The value of the integrand is \((4/3)^2\) when \((a, b)\) is in the interior of the triangle with vertices \((1/4, 1/4), (1/4, 3/4), (3/4, 3/4)\) and zero elsewhere. The triangle has area \((1/2)^2/2 = 1/8\). Therefore \( P(A > B) = (1/8)(4/3)^2 = 2/9\).

3. Let \( Y = AX + B \) where \( A, B, X \) are independent Normal(0, 1) random variables.

(a) What is \( \text{Var}[E[Y \mid X]] \)?

**Solution:** By linearity of expectation, \( E[AX + B \mid X] = E[A]X + E[B] = 0 \) so \( \text{Var}[E[Y \mid X]] = 0 \).

(b) What is \( E[\text{Var}[Y \mid X]] \)?

**Solution:** By independence, \( \text{Var}[AX + B \mid X] = \text{Var}[AX \mid X] + \text{Var}[B] = X^2 \text{Var}[A] + \text{Var}[B] = X^2 + 1 \), so \( E[\text{Var}[Y \mid X]] = E[X^2 + 1] = \text{Var}[X] + 1 = 2 \).
4. Boys and girls arrive independently at a meeting point at a rate of one boy per minute and one girl per minute, respectively. Let $T$ be the first time at which both a boy and a girl have arrived.

(a) Find the cumulative distribution function (CDF) of $T$.

**Solution:** The probability that a boy has arrived by time $t$ is $1 - e^{-t}$, i.e. the CDF of an Exponential(1) random variable. The probability that a boy has arrived by time $t$ is therefore $1 - e^{-t}$, and same for a girl. The events are independent, the probability that both have arrived by time $t$ is $P(T \leq t) = (1 - e^{-t})^2$ if $t \geq 0$ and 0 if not.

(b) What is the expected value of $T$? (Hint: You don’t have to use calculus.)

**Solution:** We can write $T = T_1 + T_2$ where $T_1$ is the arrival time of the first person and $T_2$ is the arrival time of the next person of the opposite gender. As people arrive at a rate of two per minute, $T_1$ is an Exponential(2) random variable. By the memoryless property $T_2$ is an Exponential(1) random variable. Therefore $E[T] = E[T_1] + E[T_2] = 1/2 + 1 = 3/2$.

5. A deck of cards is divided into 26 pairs. Let $X$ be the number of those pairs in which both cards are of the same suit. (A deck of cards has 4 suits and each suit has 13 cards.)

(a) What is the expected value of $X$?

**Solution:** We can write $X = X_1 + \cdots + X_{26}$ where $X_i$ is 1 if the cards in the $i$-th pair are of the same suit and 0 if not. Then $E[X_i] = P(X_i = 1)$ is the probability that the $i$-th pair’s cards are of the same suit, which is $12/51$ because conditioned on the first card’s suit, there are 12 out of 51 identical choices for the second one. By linearity of expectation $E[X] = E[X_1] + \cdots + E[X_{26}] = 26 \cdot 12/51 \approx 6.118$.

(b) What is the variance of $X$?

**Solution:** The variance of $X$ is the sum of the 26 variances of $X_i$ and the 26 · 25 covariances of $X_i$ and $X_j$. The variance of $X_i$ is $v = (12/51) \cdot (1 - 12/51) \approx 0.1799$. The covariance of $X_i$ and $X_j$ is

$$E[X_iX_j] - E[X_i] E[X_j] = P(X_i = 1, X_j = 1) - P(X_i = 1) P(X_j = 1).$$

The term $P(X_i = 1, X_j = 1)$ is the probability of the event $A$ that within both the $i$-th pair and the $j$-th pair, both cards are of the same suit. We can calculate this using the total probability theorem applied to the event $E$ that the first card of the $i$-th pair and the first card of the $j$-th pair are of the same suit:

$$P(X_i = 1, X_j = 1) = P(A) = P(A|E) P(E) + P(A|E^c) P(E^c).$$

The probability of $E$ is $12/51$. Conditioned on $E$, $A$ happens if the second cards of both pairs are also of the same suit, which is $11/50 \cdot 10/49$. Conditioned on $E^c$—for example, if the $i$-th pair’s first card is a heart and the $j$-th pair first card is a spade—$A$ happens if the second cards are a heart and a spade respectively, which happens with probability $12/50 \cdot 12/49$, and so

$$P(A) = \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{12}{51} + \frac{12}{50} \cdot \frac{12}{49} \cdot \left(1 - \frac{12}{51}\right).$$

Therefore the covariance of $X_i$ and $X_j$ equals

$$c = P(A) - \left(\frac{12}{51}\right)^2 \approx 0.0001469.$$
Finally, $\text{Var}[X] = 26 \cdot v + 26 \cdot 25 \cdot c \approx 4.7737$. 