Please turn in your solution in class on Tuesday February 2. You are encouraged to collaborate on the homework and ask for assistance, but you are required to write your own solutions, list your collaborators, acknowledge any sources of help, and provide external references if you have used any.

All circuits in this assignment are of unbounded fan-in.

**Question 1**

Recall the function \( \text{DISTINCT}: \{0, 1\}^{2n} \rightarrow \{0, 1\} \) from Lecture 1:

\[
\text{DISTINCT}(x, y) = (x_1 \neq y_1) \lor \cdots \lor (x_n \neq y_n).
\]

(a) Show that \( \text{DISTINCT} \) has a decision tree of size \( O(2^n) \).

(b) Show that the function \( \text{EQUAL} = \text{NOT \ DISTINCT} \) requires DNFs of size \( 2^n \), and therefore also decision trees of size \( 2^n \).

(c) Show that every size \( s \) DNF has a decision tree of size \( O(n^s) \).

(d) Show that the DNF

\[
x_{11}x_{12} \cdots x_{1w} \lor x_{21}x_{22} \cdots x_{2w} \lor \cdots \lor x_{s1}x_{s2} \cdots x_{sw}
\]

where \( ws = n \) requires decision tree size \( (n/s)^s \).

(e) \textbf{(Extra credit)} Let \( f(s, n) \) be the largest possible size of the smallest decision tree among those that compute DNFs of size \( n \) on \( s \) variables. From Lecture 1 and parts (a)-(e) of this exercise we know that

\[
\max\{2^{s/2}, (n/s)^s\} \leq f(s, n) \leq \min\{2^n, O(n^s)\}.
\]

Can you improve either of these bounds?

**Question 2**

(a) Show that \( \text{PARITY} \) on \( n \) bits requires DNFs of width \( n \) and DNFs of size \( 2^{n-1} \).

(b) Show that \( \text{MAJORITY} \) on \( n \) bits, \( n \) odd, requires DNFs of width \( (n + 1)/2 \) and DNFs of size \( \left(\begin{array}{c} n \\ (n+1)/2 \end{array}\right) \).

(c) Show that \( \text{PARITY} \) on \( n \) bits has a depth 3 circuit of size \( 2^{O(\sqrt{n})} \).

You may want to use the following fact: If \( f \) has an AND/OR/NOT size \( s \), depth \( d \) circuit in which the NOT gates can be in any layer but are not counted towards the depth then it has an AND/OR size \( 2s \) depth \( d \) circuit in which only negations of literals are allowed.

(d) \textbf{(Extra credit)} Show that \( \text{MAJORITY} \) on \( n \) bits has a depth 3 circuit of size \( 2^{O(\sqrt{n \log n})} \).
Question 3

(a) Show that PARITY on \( n \) bits can be computed by depth 3 circuits of size \( O(n) \) with AND, OR, and MAJORITY gates in which all the MAJORITY gates are in the same layer.

(b) Use part (a) to show that MAJORITY on \( n \) bits requires depth 3 circuits of size \( 2^{\Omega(n^{1/3})} \). (You can try for the weaker bound \( 2^{\Omega(n^{1/4})} \) first.)

(c) Use the switching lemma to show that MAJORITY requires depth 3 circuits of size \( 2^{\Omega(\sqrt{n/\log n})} \).

Question 4

Consider the following random circuit \( C \) over inputs \( x_1, \ldots, x_n \). Let \( A \) be an AND of \( a \) independent literals chosen uniformly among \( \{x_1, \ldots, x_n\} \), \( B \) be an OR of \( b \) independent copies of \( A \), and \( C \) be an AND of \( c \) independent copies of \( B \).

(a) Let \( x \in \{0, 1\}^n \) be any fixed input with at least \( 2n/3 \) ones. Show that \( C(x) \) equals zero with probability at most \( p = c \cdot \exp(-b \cdot (2/3)^a) \).

(b) Now let \( x \in \{0, 1\}^n \) be any fixed input with at most \( n/3 \) ones. Show that \( C(x) \) equals one with probability at most \( q = (b \cdot (1/3)^a)^c \).

(c) Show that for every sufficiently large \( n \) there exist \( a, b, c \) of magnitude at most polynomial in \( n \) such that \( p < 2^{-n} \) and \( q < 2^{-n} \).

(d) Show that there with nonzero probability over the choice of \( C \), for all inputs \( x \),

\[
C(x) = \begin{cases} 
1, & \text{if } x \text{ has at least } 2n/3 \text{ ones,} \\
0, & \text{if } x \text{ has at most } n/3 \text{ ones.}
\end{cases}
\]