Problem 1

Consider the following dictatorship test:

Given a function $f: \{0, 1\}^n \rightarrow \{0, 1\}$:

- Apply the linearity test to $f$:
  - Choose random $a, b \sim \{0, 1\}^n$ and reject if $f(a) + f(b) \neq f(a + b)$.
  - Choose random $x, y \sim \{0, 1\}^n$ and a random partition $(I, J)$ of $\{1, \ldots, n\}$.
    - If $f(x_I x_J) \neq f(x_I y_J)$ and $f(x_I x_J) \neq f(y_I x_J)$, reject.
    - Otherwise, accept.

A random partition $(I, J)$ of $\{1, \ldots, n\}$ is chosen by including each element in $I$ independently and uniformly at random and setting $J$ to be the complement of $I$. The notation $z_I w_J$ is used for a string in $\{0, 1\}^n$ whose $i$th coordinate is $z_i$ if $i \in I$ and $w_i$ if $i \in J$.

(a) Show that if $f$ is a dictator, i.e. $f(x) = x_i$ for some $i \in \{1, \ldots, n\}$, then the test accepts $f$.

(b) Show that if $f$ is balanced (i.e. $E_{x \sim \{0, 1\}^n}[f(x)] = 1/2$) and the test accepts $f$ with probability $1 - \delta$, then there exists a dictator $x_i$ such that $Pr_{x \sim \{0, 1\}^n}[f(x) = x_i] = 1 - O(\delta)$.

Problem 2

The double cover of a graph $G$ is the graph $G_2$ that has two copies $v_1, v_2$ of every vertex $v$ in $G$ and an edge $(v_1, v_2)$ for every edge $(v, w)$ of $G$. For example the double cover of the 3-cycle is the 6-cycle. Show that $\lambda_2(G_2) = \max(\lambda_2(G), -\lambda_n(G))$. Here $\lambda_2$ and $\lambda_n$ denote the second smallest and smallest eigenvalues of the corresponding graph.

Problem 3

Let $D \subseteq \{0, 1\}^n$ be an $\varepsilon$-biased distribution and $G$ be a regular graph whose vertices are labeled by samples of $D$ so that the number of vertices labeled $x$ is proportional to the probability of $x$ under $D$. Let $D'$ be the following distribution: Uniformly choose a random edge $(x_1, x_2)$ of $G$ and output $x_1 + x_2$. Show that $D'$ is $(\varepsilon^2 + \lambda)$-biased, where $\lambda = \max(\lambda_2(G), -\lambda_n(G))$. 