Exercises for CSCI5010

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Problem 1*. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an affine transformation. Given a set P of points in $\mathbb{R}^d$, define by $T(P) = \{T(p) \mid p \in P\}$, namely, $T(P)$ is the set of points obtained by applying the affine transformation T to P. Prove: if $R \subseteq P$ is an ϵ-kernel of P, then $T(R)$ is an ϵ-kernel of $T(P)$.

Hint: Given a directional vector $\boldsymbol{u}$, the width of P at direction $\boldsymbol{u}$ can be calculated as

$$W_{\boldsymbol{u}}(P) = \left(\max_{\boldsymbol{p} \in P} \boldsymbol{u} \cdot \boldsymbol{p}\right) - \left(\min_{\boldsymbol{p} \in P} \boldsymbol{u} \cdot \boldsymbol{p}\right)$$

where $\boldsymbol{u} \cdot \boldsymbol{p}$ is the dot product of vectors $\boldsymbol{u}$ and $\boldsymbol{p}$. To prove the claim, use your knowledge from linear algebra to figure out how a dot product would change under an affine transformation. Recall that an affine transformation is: $T(p) = \mathbf{A}\boldsymbol{p} + \boldsymbol{b}$ where $\mathbf{A}$ is a $d \times d$ matrix, and both $\boldsymbol{p}$ and $\boldsymbol{b}$ are $d \times 1$ vectors.

Problem 2* ($(1-\epsilon)$-Approximate Top-1 Search). Let P be a set of points in $\mathbb{R}^d$ where d is a constant, and each point has a positive coordinate on every dimension. We will view each point $p \in P$ as a d-dimensional vector $\boldsymbol{p} = (p[1], p[2], ..., p[d])$ where $p[i]$ $(1 \le i \le d)$ is the i-th coordinate of p. Given a directional vector $\boldsymbol{u}$ where $u[i] \ge 0$ for each $i \in [d]$, define

$$\text{top}_{\boldsymbol{u}}(P) = \max_{\boldsymbol{p} \in P} \boldsymbol{u} \cdot \boldsymbol{p}$$

where $\boldsymbol{u} \cdot \boldsymbol{p}$ is the dot product of vectors $\boldsymbol{u}$ and $\boldsymbol{p}$. Given $0 < \epsilon < 1$, describe an algorithm that computes in $O(n)$ expected time a subset $R \in P$ such that

- $|R| = O(1/\epsilon^d)$, and
- for any directional vector $\boldsymbol{u}$, it holds that $\text{top}_{\boldsymbol{u}}(R) \ge (1-\epsilon) \cdot \text{top}_{\boldsymbol{u}}(P)$.

Hint: Add the origin to P.

Problem 3. Prove the order-reversal property of dual transformation.

Problem 4. Prove the intersection preserving property of dual transformation.

Problem 5. Let ℓ_1 and ℓ_2 be two parallel non-vertical lines in the primal space $\mathbb{R}^2$. Prove: their vertical distance equals the distance of points ℓ_1^* and ℓ_2^* in the dual space.

Problem 6. Let A, B, C, and D be four points in the primal space $\mathbb{R}^2$ that have distinct x coordinates. Suppose that triangle ABC has an area smaller than ABD. Let ℓ be the line passing points A and B in the primal space. Prove: in the dual space, point ℓ^* has a smaller vertical distance to line C^* than to line D^*.

Note: The vertical distance from a point (a, b) to a line $y = c_1 x - c_2$ equals $|b - (c_1 \cdot a - c_2)|$.