Exercises for CSCI5010

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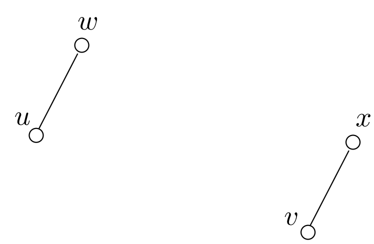
Let P be a set of n points in $\mathbb{R}^d$, where d is a constant. Denote by T the quadtree of P. In the lecture, we proved that our s-WSPD algorithm computes an s-WSPD of P with $O(s^d \cdot n \cdot h)$ pairs, where h is the height of T. Now, apply the same algorithm on the *compressed* quadtree tree T_{com} of P. In this exercise, you will prove that the algorithm produces an s-WSPD of $O(s^d \cdot n)$ pairs.

We will apply the same charging strategy as introduced in the lecture. Every time the algorithm generates $\{u, v\}$ from $\{w, v\}$ by splitting w (i.e., w is the parent of u in T_{com}), we charge the pair $\{u, v\}$ on w.



Solve the following problems.

Problem 1. For each node z in T_{com}, we use *level*(z) to denote the level of z in the original quadtree T. Prove: *level*$(v) \geq$ *level*$(w) \geq$ *level*(x).

Remark: Recall that if a node is at level ℓ of T, the node corresponds to a box with side length $1/2^\ell$ on each dimension. Essentially, you need to prove that the box of v is no larger than that of w, which in turn is no larger than that of x.

Hint: Our algorithm always splits the "larger" node in a pair.

Problem 2. Fix a node w in T_{com} and a child u of w. Prove: there are $O(s^d)$ nodes v in T_{com} satisfying (i) *level*$(w) =$ *level*(v) and (ii) w is charged for the pair $\{u, v\}$.

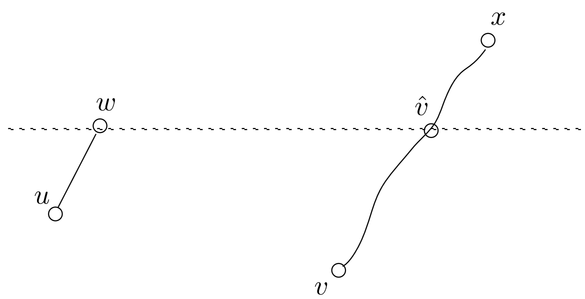
Problem 3. Fix a node w in T_{com} and a child u of w. Prove: there are $O(s^d)$ nodes v in T_{com} satisfying

- *level*$(w) =$ *level*(x) where x is the parent of v in T_{com} and
- w is charged for the pair $\{u, v\}$.

Problem 4. Fix a node w in T_{com} and a child u of w. Let S be the collection of nodes v of T_{com} satisfying

- *level*$(v) >$ *level*$(w) >$ *level*(x) where x is the parent of v in T_{com} and
- w is charged for the pair $\{u, v\}$.

Consider any node $v \in S$ and let x be the parent of v in T_{com}. Identify the node in T (the original quadtree) at level *level*(w) on the path from x to v in T. We will refer to $\hat{v}$ the *anchor node of v with respect to w*. Note that $\hat{v}$ has only a single child and does not exist in T_{com} (i.e., $\hat{v}$ is removed by compression).



Prove: the nodes in S have distinct anchor nodes with respect to w.

Hint: Which nodes on the path from x to v in T have only one child?

Problem 5. Fix a node w in T_{com} and a child u of w. Let S be the collection of nodes v of T_{com} satisfying

- $level(v) > level(w) > level(x)$ where x is the parent of v in T_{com} and
- w is charged for the pair $\{u, v\}$.

Prove: $|S| = O(s^d)$.

Hint: Apply the Packing Lemma to bound the number of anchor nodes.

Problem 6. Prove: Each node w of T_{com} can be charged only $O(s^d)$ times.