Further Insights into SCCs

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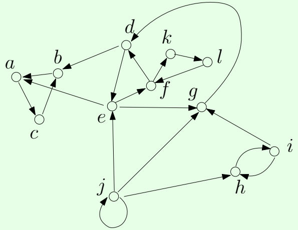
Given a directed graph $G = (V, E)$, the goal of the **strongly connected components problem** is to divide V into disjoint subsets, each being an SCC.

Example:



We should output: $\{a, b, c\}$, $\{d, e, f, g, k, l\}$, $\{h, i\}$, and $\{j\}$.

Algorithm

Step 1: Run DFS on G and list the vertices by the order they turn black.

- If a vertex is the i-th vertex turning black, define its **label** as i.

Step 2: Obtain the **reverse graph** G^{rev} by flipping all the edge directions in G.

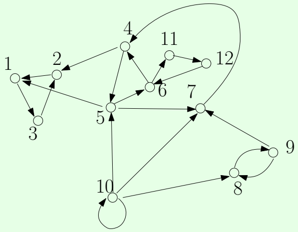
Step 3: Perform DFS on G^{rev} subject to the following rules:

- **Rule 1:** Start at the vertex with the largest label.
- **Rule 2:** When a restart is needed, do so from the white vertex with the largest label.

Output the vertices in each DFS-tree as an SCC.

Next, we will show how to implement the SCC algorithm in $O(|V| + |E|)$ time. You can assume that $V = \{1, 2, ..., n\}$.

Example:

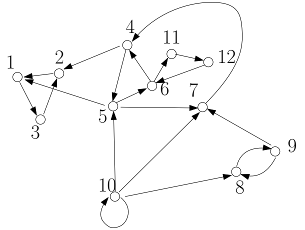


G

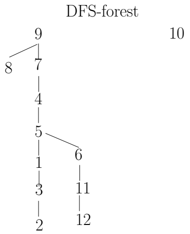
Step 1

Perform DFS on G and record the turn-black order in an array A.

- $A[i]$ stores the vertex with label i.



G



vertex with label 1 — vertex with label 2

8	2	3	1	12	11	6	5	4	7	9	10

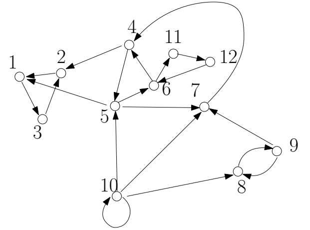
A

Time: $O(|V| + |E|)$.

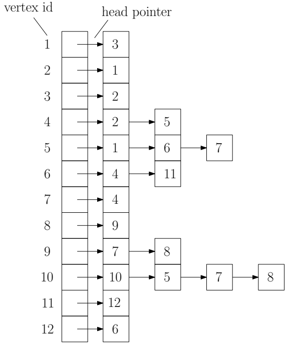
Step 2

Obtain $G^{rev} = (V, E^{rev})$ from G in $O(|V| + |E|)$ time.

We will illustrate how to do so through an example.



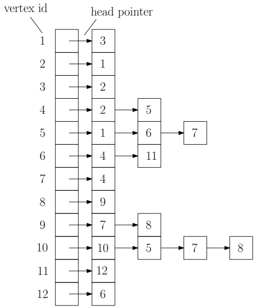
G



adjacency list of G

Step 2

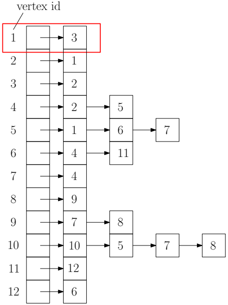
Initialize the head-pointer array for G^{rev}.



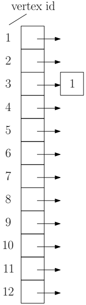
adj. list of G adj. list of G^{rev}

Step 2

Scan the neighbor list of each $u \in V$ in G.
For every out-neighbor v of u, add u to the neighbor list of v in G^{rev}.



adj. list of G



adj. list of G^{rev}

Step 2

Scan the neighbor list of each $u \in V$ in G.
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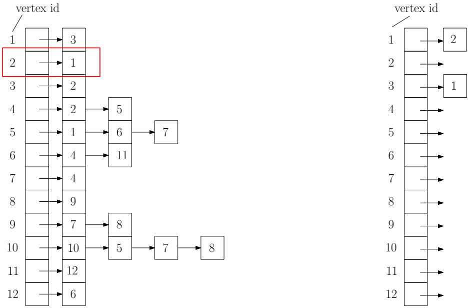

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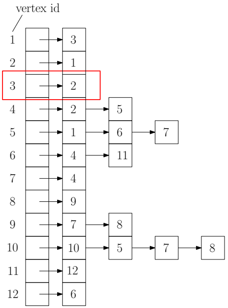
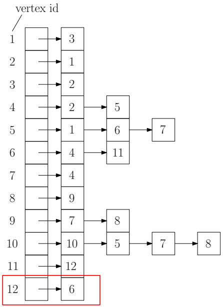


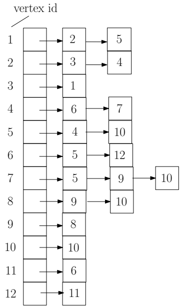
adj. list of G adj. list of G^{rev}

Step 2

Scan the neighbor list of each $u \in V$ in G.
For every out-neighbor v of u, add u to the neighbor list of v in G^{rev}.



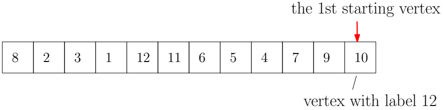
adj. list of G



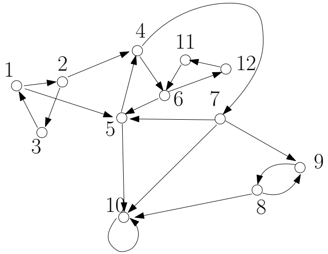
adj. list of G^{rev}

Step 3

Perform DFS on G^{rev} and use A to select the vertex to start/restart from.



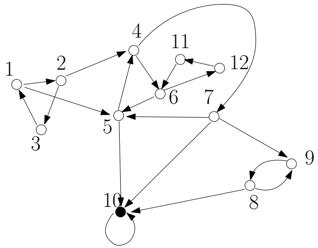
A



G^{rev}

Step 3

Start the 1st DFS on G^{rev} from vertex 10. Output $\{10\}$.



G^{rev}

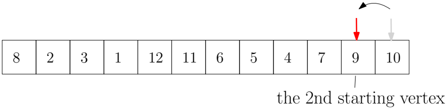
DFS-tree

10

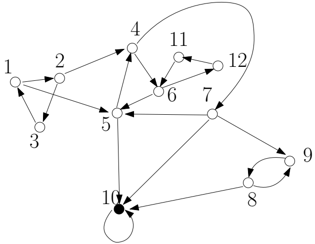
Vertex 10 is now black.

Step 3

Scan A backwards from $A[12]$ and find the first white vertex $A[11] = 9$.



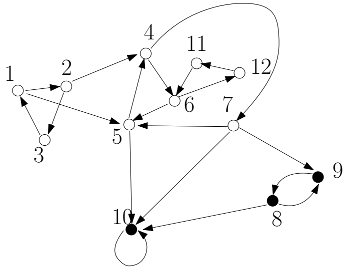
A



G^{rev}

Step 3

Start the 2rd DFS on G^{rev} from 9. Output $\{8, 9\}$.



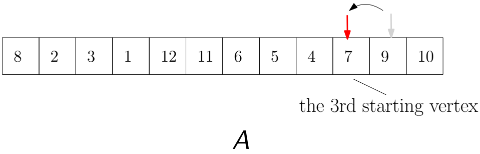
G^{rev}

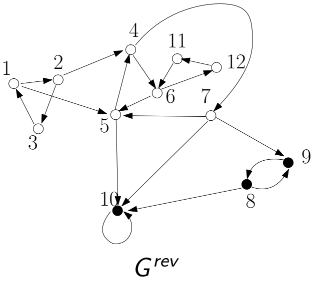
DFS-tree

Vertices 8 and 9 are now black.

Step 3

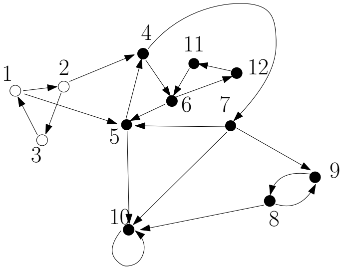
Scan A backwards from $A[11]$ and find the first white vertex $A[10] = 7$.



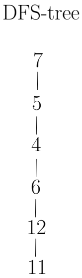


Step 3

Start the 3rd DFS on G^{rev} from 7. Output $\{7, 5, 4, 6, 12, 11\}$.



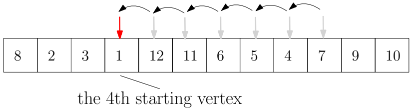
G^{rev}



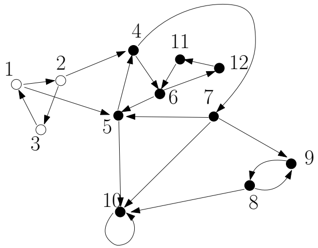
Vertices 7, 5, 4, 6, 12, and 11 are now black.

Step 3

Scan A backwards from $A[10]$ and find the first white vertex $A[4] = 1$.



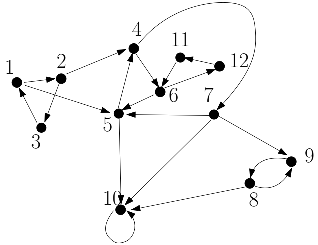
A



G^{rev}

Step 3

Start the 4th DFS on G^{rev} from 1. Output $\{1, 2, 3\}$.



G^{rev}

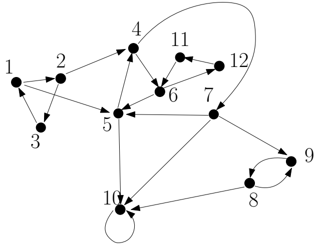
DFS-tree

Step 3

Scan A backwards from 1 and find no other white vertices. The algorithm finishes.

8	2	3	1	12	11	6	5	4	7	9	10

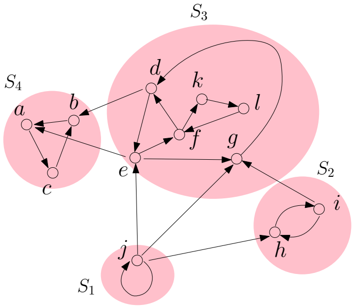
A

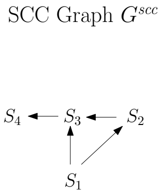


G^{rev}

Next, we will unveil a mathematical structure of the SCC problem that suggests a generic algorithmic paradigm.

Sink SCC





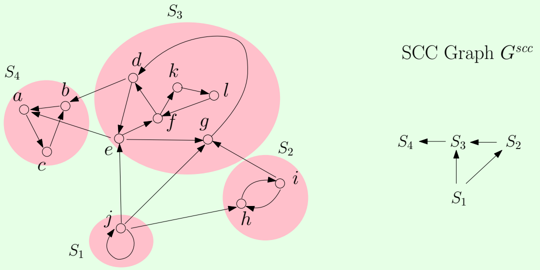
An SCC is a **sink SCC** if it has no outgoing edge in G^{scc}.

S_4 is the only sink SCC in the above example.

A conceptual SCC strategy

1. **while** G^{scc} not empty **do**
2. $\quad$ $S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in S
4. $\quad$ remove all the vertices in S from G;
 $\quad$ delete vertex S from G^{scc}

Example:



DFS from anywhere in S_4 finds SCC $\{a, b, c\}$.

A conceptual SCC strategy

1. **while** G^{scc} not empty **do**
2. $\quad$ $S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in S
4. $\quad$ remove all the vertices in S from G;
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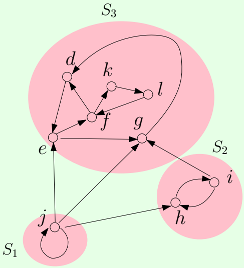
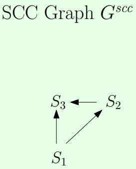
Example:



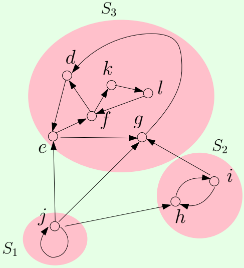


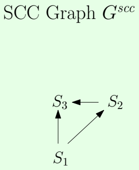
Delete S_4 from G and G^{scc}. New sink vertex: S_3.

A conceptual SCC strategy

1. **while** G^{scc} not empty **do**
2. $\quad S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in S
4. $\quad$ remove all the vertices in S from G;
   delete vertex S from G^{scc}

Example:



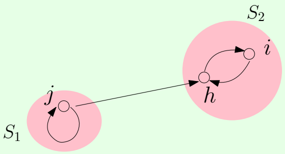


DFS from anywhere in S_3 finds SCC $\{d, e, f, g, k, l\}$.

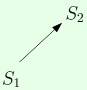
A conceptual SCC strategy

1. **while** G^{scc} not empty **do**
2. $S \leftarrow$ a sink SCC
3. run DFS from any vertex in S
4. remove all the vertices in S from G; delete vertex S from G^{scc}

Example:



SCC Graph G^{scc}

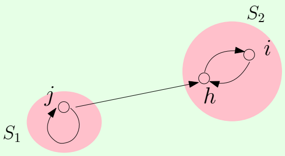


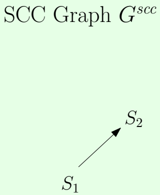
Delete S_3. New sink vertex: S_2.

A conceptual SCC strategy

1. **while** G^{scc} not empty **do**
2. $\quad S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in S
4. $\quad$ remove all the vertices in S from G; delete vertex S from G^{scc}

Example:



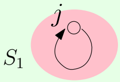


DFS from anywhere in S_2 finds SCC $\{i, h\}$.

A conceptual SCC strategy

while G^{scc} not empty **do**
    $S \leftarrow$ a sink SCC
    run DFS from any vertex in S
    remove all the vertices in S from G;
    delete vertex S from G^{scc}

Example:



SCC Graph G^{scc}

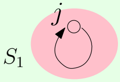
S_1

Delete S_2. New sink vertex S_1.

A conceptual SCC strategy

1. **while** G^{scc} not empty **do**
2. $\quad$ $S \leftarrow$ a sink SCC
3. $\quad$ run DFS from any vertex in S
4. $\quad$ remove all the vertices in S from G; delete vertex S from G^{scc}

Example:



SCC Graph G^{scc}

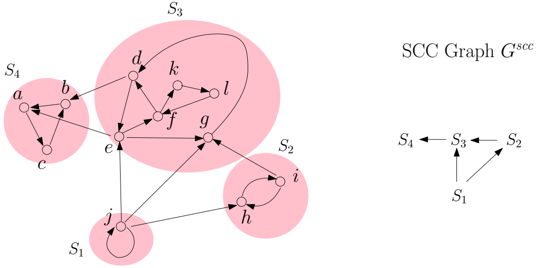
S_1

The 4th DFS finds SCC $\{j\}$.

Question:

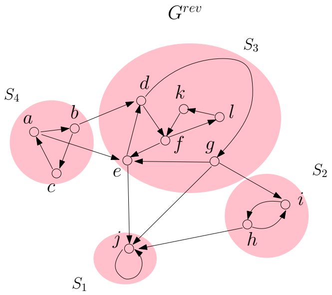
Why does our SCC algorithm work on the **reverse** graph, as opposed to the **original** one?

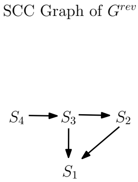
Answer: Non-trivial to find the next sink SCC.



Not easy: You need to find a vertex in S_4 first, then a vertex in S_3, then one in S_2, and finally in S_1.

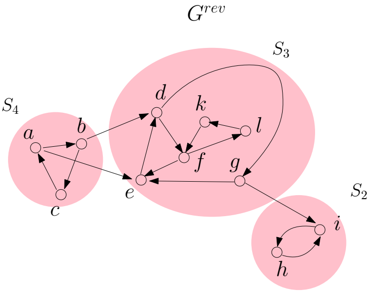
It turns out that finding the next sink SCC on the **reverse** graph is **much** easier.

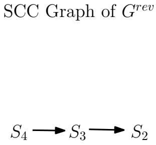




Sink SCC $= S_1$.
DFS from j finds SCC $\{j\}$

It turns out that finding the next sink SCC on the reverse graph is **much** easier.

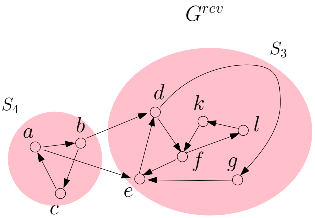


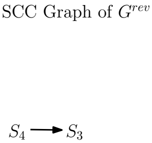


Sink SCC = S_2.

DFS from anywhere in S_2 finds SCC $\{h, i\}$

It turns out that finding the next sink SCC on the reverse graph is **much** easier.



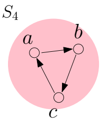


Sink SCC $= S_3$.

DFS from anywhere in S_3 finds SCC $\{d, e, f, g, k, l\}$.

It turns out that finding the next sink SCC on the reverse graph is **much** easier.

G^{rev}



SCC Graph of G^{rev}

S_4

Sink SCC $= S_4$. The last DFS finds SCC $\{a, b, c\}$.

This is exactly how our SCC algorithm finds the SCCs.