Some Exercises on the “Three Basic Techniques”

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You have learned three basic techniques in algorithm design:

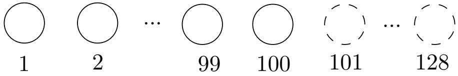
- Recursion
- Repeating (till success)
- Geometric Series.

In this tutorial, we will discuss some exercises that can be solved using these techniques.

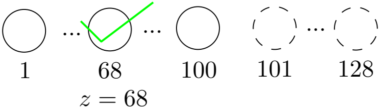
Exercise 1

Recall that our RAM model has an atomic operation RANDOM(x, y) which, given integers x, y, returns an integer chosen uniformly at random from $[x, y]$.

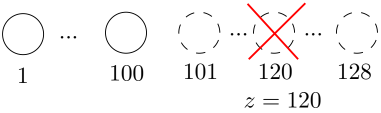
Suppose that you are allowed to call the operation only with $x = 1$ and $y = 128$. Describe an algorithm to obtain a uniformly random number between 1 and 100. Your algorithm must finish in $O(1)$ expected time.



Call RANDOM(1,128) and let z be its return value. Output z if it is in $[1, 100]$.



Otherwise, repeat from the beginning.



We need to call the operator at most twice in expectation because each time z has probability $100/128$ to fall in the range we want. Therefore, our algorithm finishes in $O(1)$ expected time.

Exercise 2

Suppose that we enforce a harder constraint that you are allowed to call RANDOM(x, y) only with $x = 0$ and $y = 1$. Describe an algorithm to generate a uniformly random number in $[1, n]$ for an arbitrary integer n. Your algorithm must finish in $O(\log n)$ expected time.

Suppose n is a power of 2; then how can we use recursion to solve this problem?

1. Set $z =$RANDOM(x, y).
2. If $z = 0$, we have a subproblem: generate a uniformly random number in the first half of the range;
   If $z = 1$, we have a subproblem: generate a uniformly random number in the second half of the range.

Considering the subproblem solved, we finish the algorithm.

Analysis of the Algorithm

$$f(1) = O(1)$$
$$f(n) \leq f(n/2) + O(1) \quad \text{, for } n > 1$$

Thus, we have

$$f(n) = O(\log n)$$

Think: Why does the algorithm require n to be a power of 2?

Next, we will extend our algorithm to support values of n that are not powers of 2.

First, obtain the smallest power of 2 that is at least n.

- Try $1, 2, 4, ...$, until reaching m such that $n \leq m < 2n$. This takes $O(\log n)$ time.

We have known how to generate a uniformly random number y in $[1, m]$ in $O(\log n)$ time.

If $y \leq n$, return y; otherwise, repeat the algorithm. At most 2 repeats are needed in expectation. The overall time is there $O(\log n)$ in expectation.

Exercise 3

Recall the k-selection problem:

> You are given a set S of n integers in an array and an integer $k \in [1, n]$. Find the k-th smallest integer of S.

Suppose there is a deterministic algorithm $\mathcal{A}_1$ which returns the median of n integers in $O(n)$ time. Can you use $\mathcal{A}_1$ as a blackbox to solve k-selection in $O(n)$ time?

Consider the following algorithm.

1. Get the median v of S from $\mathcal{A}_1(S)$.
2. Divide S into S_1 and S_2 where
    - $S_1 =$ the set of elements in S less than or equal to v;
    - $S_2 =$ the set of elements in S greater than v.
3. If $|S_1| \geq k$, then return $S' = S_1$ and $k' = k$; else return $S' = S_2$ and $k' = k - |S_1|$

Since $\mathcal{A}_1$ is deterministic, we always succeed in obtaining a subproblem with size no larger than $\lceil \frac{|S|}{2} \rceil$.

Analysis of the Algorithm

$$f(1) = O(1)$$
$$f(n) \leq f(n/2) + O(n)$$

Thus, $f(n) = O(n)$.

What if $\mathcal{A}_1$ returns the $\lceil \frac{4}{5} n \rceil$-th smallest integer of n integers in $O(n)$ time. Can you still use $\mathcal{A}_1$ as a blackbox to solve k-selection in $O(n)$ time?

Instead of shrinking the size of subproblem by half, we shrink it by $\frac{4}{5}$.

We can still use $\mathcal{A}_1$ to shrink the problem size by a constant factor. From the geometric series we know that the total cost will be $O(n)$.

Think: If $\mathcal{A}_1$ returns the $\lceil \frac{99}{100} n \rceil$-th smallest integer of n integers in $O(n)$ time, can you still use $\mathcal{A}_1$ as a blackbox to solve k-selection in $O(n)$ time?

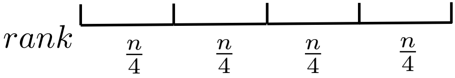
Exercise 4

Let's still focus on the k-selection problem. In the lecture, we shrink the input size of the subproblem into at most $\frac{2}{3}n$. Now, we want to shrink the input size into at most $\frac{n}{2}$. Give an algorithm to achieve the purpose in $O(n)$ expected time.

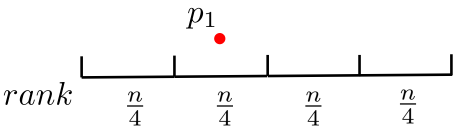
A simple solution: run our "$\frac{2n}{3}$-algorithm" twice. The number of remaining elements becomes at most $\frac{4n}{9}$.

Next, let us look at another way to achieve the purpose, assuming for simplicity that n is a multiple of 4.

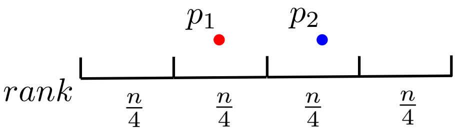
First, divide the rank space into 4 equal partitions.



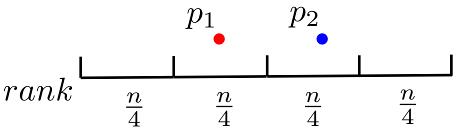
Second, take an element p_1 from S uniformly at random. Repeat until $rank(p_1)$ is in range $[\frac{n}{4}, \frac{n}{2}]$.



Third, take an element p_2 from S uniformly at random. Repeat until $rank(p_2)$ is in range $[\frac{1}{2}n, \frac{3}{4}n]$.



- If $k \leq rank(p_1)$, set S' = the set of elements in S less than or equal to p_1, $k' = k$.
- If $rank(p_1) < k < rank(p_2)$, set S' = the set of elements in S larger than p_1 and smaller than p_2, $k' = k - rank(p_1)$.
- If $k \geq rank(p_2)$, set S' = the set of elements in S larger than or equal to p_2, $k' = k - rank(p_2)$.



In any case, we have $|S'| \leq \frac{n}{4} + \frac{n}{4} = \frac{n}{2}$.

In expectation, 4 repeats are needed for p_1, and 4 repeats for p_2 (think: why?).