CSCI3160: Quiz 2

Name: Student ID

Problem 1 (50%). Consider the optimal BST problem on $S = \{1, 2, 3, 4\}$ and the weight array $W = (25, 15, 20, 50)$. Given integers $a, b \in [1, 4]$, define

$$optavg(a, b) = \begin{cases} 0 & \text{if } a > b \\ \text{the smallest average cost of a BST on } \{a, a+1, ..., b\} & \text{if } a \leq b \end{cases}$$

Some function values have been calculated for you:

$$\begin{aligned} optavg(1,1) &= 25 \\ optavg(1,2) &= 55 \\ optavg(1,3) &= 105 \\ optavg(2,4) &= 135 \\ optavg(3,4) &= 90 \\ optavg(4,4) &= 50. \end{aligned}$$

Prove: The optimal BST is not unique.

Solution. As derived in the lecture:

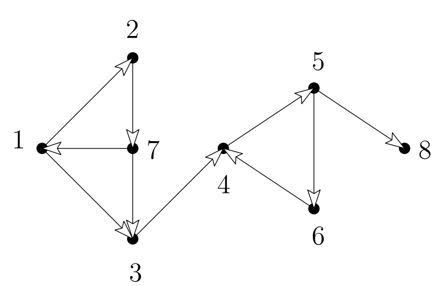
$$optavg(1,4) = \left(\sum_{i=1}^{4} W[i]\right) + \min_{r=1}^{4} \{optavg(1, r-1) + optavg(r+1, b)\}.$$

We enumerate all possibilities of the root's key:

- If the root has key 1, the best BST has average cost $110 + optavg(2,4) = 110 + 135 = 245$.
- If the root has key 2, the best BST has average cost $110 + optavg(1,1) + optavg(3,4) = 110 + 25 + 90 = 225$.
- If the root has key 3, the best BST has average cost $110 + optavg(1,2) + optavg(4,4) = 110 + 55 + 50 = 215$.
- If the root has key 4, the best BST has average cost $110 + optavg(1,3) = 110 + 105 = 215$.

It thus follows that an optimal BST has average cost 215. As setting the root key to 3 or 4 can both yield an optimal BST, we know that there are at least two optimal BSTs.

Problem 2 (50%). Consider the directed graph G below.



Run the SCC (strongly connected components) algorithm taught in our lecture on this graph. Recall that the algorithm performs 3 steps:

- Step 1: Run DFS on G.
- Step 2: Create a new graph G'.
- Step 3: Run DFS on G'.

You must start the DFS of Step 1 from vertex 1. Answer the following questions:

(i) Indicate the vertices' turn-black order obtained in Step 1.

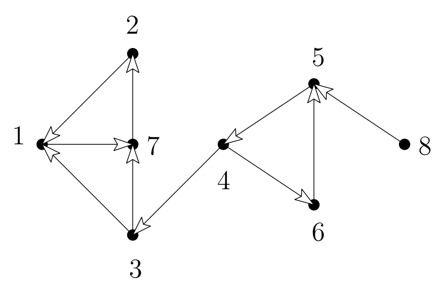
(ii) Draw the graph G'.

(iii) Indicate all the SCCs output by Step 3, and the root of each DFS-tree produced in this step.

Solution. (i) The following are both correct turn-black orders:

- 6, 8, 5, 4, 3, 7, 2, 1
- 8, 6, 5, 4, 3, 7, 2, 1

(ii)



(iii) This solution holds for both turn-black orders given in (i).
First SCC: $\{1, 7, 2\}$, root vertex 1.
Second SCC: $\{3\}$, root vertex 3.
Third SCC: $\{4, 5, 6\}$, root vertex 4.
Fourth SCC: $\{8\}$, root vertex 8.