Approximation Algorithms 3: Set Cover and Hitting Set

Yufei Tao

Department of Computer Science and Engineering
Chinese University of Hong Kong

Set Cover

We are given a collection $\mathcal{S}$ where each member of $\mathcal{S}$ comes from a certain domain (which is not important).

Define the **universe** $U = \bigcup_{S \in \mathcal{S}} S$.

A sub-collection $\mathcal{C} \subseteq \mathcal{S}$ is a **set cover** (of U) if every element of U appears in at least one set in $\mathcal{C}$.

The set cover problem:
Find a set cover with the smallest size.

Example: $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $\mathcal{S} = \{S_1, S_2, ..., S_5\}$ where

$$\begin{aligned} S_1 &= \{1, 2, 3, 4\} \\ S_2 &= \{2, 5, 7\} \\ S_3 &= \{6, 7\} \\ S_4 &= \{1, 8\} \\ S_5 &= \{1, 2, 3, 8\}. \end{aligned}$$

An optimal solution is $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$.

The input size of the set cover problem is $n = \sum_{S \in \mathcal{S}} |S|$.

The problem is NP-hard.

- No one has found an algorithm solving the problem in time polynomial in n.
- Such algorithms cannot exist if $\mathcal{P} \neq \mathcal{NP}$.

$\mathcal{A}$ = an algorithm that, given any legal input $\mathcal{S}$ with universe U, returns a set cover $\mathcal{C}$.

Denote by $OPT_{\mathcal{S}}$ the smallest size of all set covers when the input collection is $\mathcal{S}$.

> $\mathcal{A}$ is a ρ-**approximate algorithm** for the set cover problem if, for any legal input $\mathcal{S}$, $\mathcal{A}$ can return a set cover with size at most $\rho \cdot OPT_{\mathcal{S}}$.

The value ρ is the **approximation ratio**.
We say that $\mathcal{A}$ achieves an approximation ratio of ρ.

Consider the following algorithm.

Input: A collection $\mathcal{S}$

1. $\mathcal{C} = \emptyset$
2. **while** U still has elements not covered by any set in $\mathcal{C}$
3. $F \leftarrow$ the set of elements in U not covered by any set in $\mathcal{C}$
   /* for each set $S \in \mathcal{S}$, define its **benefit** to be $|S \cap F|$ */
4. add to $\mathcal{C}$ a set in $\mathcal{S}$ with the largest benefit
5. **return** $\mathcal{C}$

It is easy to show:

- The $\mathcal{C}$ returned is a set cover;
- The algorithm runs in time polynomial to n.

We will prove later that the algorithm is $(1 + \ln |U|)$-approximate.

Example: $S_1 = \{1, 2, 3, 4\}$, $S_2 = \{2, 5, 7\}$, $S_3 = \{6, 7\}$, $S_4 = \{1, 8\}$, $S_5 = \{1, 2, 3, 8\}$

- In the beginning, $\mathcal{C} = \emptyset$ and $F = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
- Next, we can add S_1 or S_5 to $\mathcal{C}$ (benefit 4). The choice is arbitrary; suppose we add S_1. Now, $F = \{5, 6, 7, 8\}$.
- Next, we can add S_2 or S_3 (benefit 2). The choice is arbitrary; suppose we add S_2. Now, $F = \{6, 8\}$.
- Next, we can add S_3, S_4, or S_5 (benefit 1). The choice is arbitrary; suppose we add S_3. Now, $F = \{8\}$.
- Next, we cab add S_4 or S_5 (benefit 1). The choice is arbitrary; suppose we add S_4. Now, $F = \emptyset$.

The algorithm terminates with $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$.

Theorem 1: The algorithm returns a set cover with size at most $1 + (\ln |U|) \cdot OPT_{\mathcal{S}} \leq (1 + \ln |U|) \cdot OPT_{\mathcal{S}}$.

$\mathcal{C}$ = the set cover returned.
$t = |\mathcal{C}|$.

Denote the sets in $\mathcal{C}$ as $S_1, S_2, ..., S_t$, picked in the order shown.

For each $i \in [1, t]$, define z_i as the size of F after S_i is picked.
Specially, define $z_0 = |U|$.

$z_t = 0$ and $z_{t-1} \geq 1$. **Think:** why?

Denote by $\mathcal{C}^*$ an optimal set cover, namely, $OPT_{\mathcal{S}} = |\mathcal{C}^*|$.

We will prove later:

Lemma 1: For $i \in [1, t]$, it holds that

$$z_i \leq z_{i-1} \cdot \left(1 - \frac{1}{OPT_{\mathcal{S}}}\right).$$

From Lemma 1, we get:

$$\begin{aligned} z_{t-1} &\le z_{t-2} \cdot \left(1 - \frac{1}{OPT_{\mathcal{S}}}\right) \\ &\le z_{t-3} \cdot \left(1 - \frac{1}{OPT_{\mathcal{S}}}\right)^2 \\ &\ldots \\ &\le z_0 \cdot \left(1 - \frac{1}{OPT_{\mathcal{S}}}\right)^{t-1} = |U| \cdot \left(1 - \frac{1}{OPT_{\mathcal{S}}}\right)^{t-1} \\ &\le |U| \cdot e^{-\frac{t-1}{OPT_{\mathcal{S}}}} \end{aligned}$$

where the last inequality used the fact $1 + x \le e^x$ for any real value x.

As $z_{t-1} \ge 1$, we have

$$1 \le |U| \cdot e^{-\frac{t-1}{OPT_{\mathcal{S}}}} \quad (1)$$

which resolves to $t \le 1 + (\ln |U|) \cdot OPT_{\mathcal{S}}$. This proves Theorem 1.

Proof of Lemma 1

Before S_i is chosen, F has z_{i-1} elements.

At this moment, at least one set $S^* \in \mathcal{C}^*$ has a benefit at least

$$\frac{z_{i-1}}{|\mathcal{C}^*|} = \frac{z_{i-1}}{OPT_{\mathcal{S}}} > 0$$

(every element of F must appear in some set in $\mathcal{C}^*$).

The set S^* cannot have been chosen (every chosen set has benefit 0) and is thus a candidate for S_i. It thus follows that S_i must have a benefit at least $\frac{z_{i-1}}{OPT_{\mathcal{S}}}$ (greedy). Therefore:

$$\begin{aligned} z_i &= |F \setminus S_i| = |F| - |F \cap S_i| \\ &\leq z_{i-1} - \frac{z_{i-1}}{OPT_{\mathcal{S}}} \\ &= z_{i-1}\left(1 - \frac{1}{OPT_{\mathcal{S}}}\right) \end{aligned}$$

Next, we will introduce a closely related problem called the **hitting set problem**.

Hitting Set

Let U be a finite set called the **universe**.

We are given a collection $\mathcal{S}$ where each member of $\mathcal{S}$ is a set $S \subseteq U$.

A subset $H \subseteq U$ **hits** a set $S \in \mathcal{S}$ if $H \cap S \neq \emptyset$.
A subset $H \subseteq U$ is a **hitting set** (of $\mathcal{S}$) if it hits all the sets in $\mathcal{S}$.

> **The hitting set problem:**
> Find a hitting set H of the minimize size.

Example: $U = \{1, 2, 3, 4, 5\}$ and $\mathcal{S} = \{S_1, S_2, ..., S_8\}$ where

$$\begin{aligned} S_1 &= \{1, 4, 5\} \\ S_2 &= \{1, 2, 5\} \\ S_3 &= \{1, 5\} \\ S_4 &= \{1\} \\ S_5 &= \{2\} \\ S_6 &= \{3\} \\ S_7 &= \{2, 3\} \\ S_8 &= \{4, 5\} \end{aligned}$$

An optimal solution is $H = \{1, 2, 3, 4\}$.

The input size of the set cover problem is $n = \sum_{S \in \mathcal{S}} |S|$.

The problem is NP-hard.

- No one has found an algorithm solving the problem in time polynomial in n.
- Such algorithms cannot exist if $\mathcal{P} \neq \mathcal{NP}$.

$\mathcal{A}$ = an algorithm that, given any legal input $\mathcal{S}$ with universe U, returns a hitting set.

Denote by $OPT_{\mathcal{S}}$ the smallest size of all hitting sets.

> $\mathcal{A}$ is a ρ-**approximate algorithm** for the hitting set problem if, for any legal input $\mathcal{S}$, $\mathcal{A}$ can return a hitting set with size at most $\rho \cdot OPT_{\mathcal{S}}$.

The value ρ is the **approximation ratio**.
We say that $\mathcal{A}$ achieves an approximation ratio of ρ.

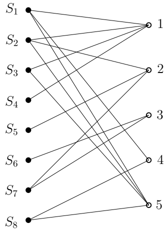
Hitting set and set cover are essentially the same problem.

Let $\mathcal{S}$ be the input to the hitting set problem (recall that $\mathcal{S}$ is a collection of sets). By converting the problem to an instance of set cover, we can obtain a polynomial-time hitting-set algorithm that guarantees an approximation ratio of

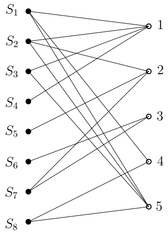
$$1 + \ln |\mathcal{S}|.$$

The proof is left as a regular exercise, but the next slide illustrates the key idea behind the conversion.

Consider the hitting set example on Slide 15. Let us create a bipartite graph G:



Each set $S \in \mathcal{S}$ corresponds to a vertex on the left of G.
Each element $e \in U$ corresponds to a vertex on the right of G.
An edge exists between vertex S and vertex e if and only if $e \in S$.



Solving the hitting set problem is equivalent to finding a smallest set R of **right** vertices such that every left vertex is adjacent to at least one vertex in R.

This gives rise to the set cover example on Slide 3.