Single Source Shortest Paths with Arbitrary Weights

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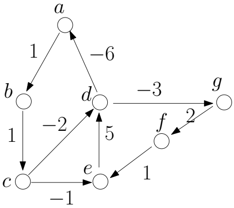
We will continue our discussion on the single source shortest path (SSSP) problem, but this time we will allow the edges to take **negative** weights.

Dijkstra's algorithm no longer works. We will learn another algorithm — called **the Bellman-Ford algorithm** — to solve the problem.

Let $G = (V, E)$ be a directed graph. Let w be a function that maps each edge in $e \in E$ to an integer $w(e)$, **which can be positive, 0, or negative**.



Shortest Path

Consider a path in G: $(v_1, v_2), (v_2, v_3), ..., (v_\ell, v_{\ell+1})$, for some integer $\ell \geq 1$. We define the path's **length** as

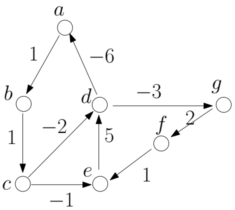
$$\sum_{i=1}^{\ell} w(v_i, v_{i+1}).$$

A **shortest path** from u to v has the minimum length among all the paths from u to v. Denote by $spdist(u, v)$ the length of a shortest path from u to v.

If v is unreachable from u, $spdist(u, v) = \infty$.

New: The length of a path can be negative!

Example



The path $c \to d \to g$ has length -5.

Can you find a shortest path from a to c? Counter-intuitively, it has an **infinite** number of edges such that $spdist(a, c) = -\infty$!

- This is due to the **negative cycle** $a \to b \to c \to d \to a$.

Negative cycle

A path $(v_1, v_2), (v_2, v_3), ..., (v_\ell, v_{\ell+1})$ is a **cycle** if $v_{\ell+1} = v_1$.

It is a **negative cycle** if its length is negative, namely:

$$\sum_{i=1}^{\ell} w(v_i, v_{i+1}) < 0$$

SSSP Problem: Let $G = (V, E)$ be a directed simple graph, where function w maps every edge of E to an arbitrary integer. **It is guaranteed that G has no negative cycles.** Given a **source vertex** s in V, we want to find a shortest path from s to t for every vertex $t \in V$ reachable from s.

The output is a **shortest path tree** T:

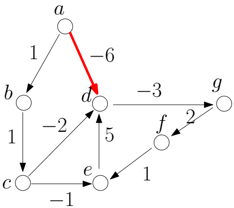
- The vertex set of T is V.
- The root of T is s.
- For each node $u \in V$, the root-to-u path of T is a shortest path from s to u in G.

We will learn an algorithm called **the Bellman-Ford algorithm** that solves both problems in $O(|V||E|)$ time.

We will focus on **computing** $spdist(s, v)$, namely, the shortest path distance from the source vertex s to every vertex $v \in V$.

Constructing the shortest paths is easy and will be left to you.

Example



This graph has no negative cycles.

Lemma: For every vertex $v \in V$, at least one shortest path from s to v is **simple path**, namely, a path where no vertex appears twice.

The proof is left to you — note that you must use the condition that no negative cycles are present.

Corollary: For every vertex $v \in V$, there is a shortest path from s to v having at most $|V| - 1$ edges.

Edge Relaxation

For every vertex $v \in V$, we will — at all times — maintain a value $dist(v)$ equal to the shortest path length from s to v **found so far**.

Relaxing an edge (u, v) means:

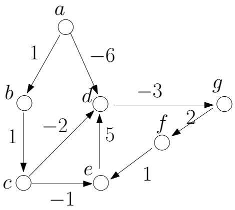
- If $dist(v) \leq dist(u) + w(u, v)$, do nothing;
- Otherwise, reduce $dist(v)$ to $dist(u) + w(u, v)$.

The Bellman-Ford algorithm

1. Set $dist(s) \leftarrow 0$, and $dist(v) \leftarrow \infty$ for all other vertices $v \in V$.
2. Repeat the following $|V| - 1$ times
   - Relax all edges in E (the relaxation order does not matter)

Example

Suppose that the source vertex is a.



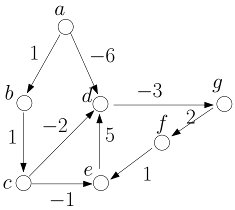
vertex v	$dist(v)$
a	0
b	∞
c	∞
d	∞
e	∞
f	∞
g	∞

For illustration purposes, we will relax the edges in alphabetic order: $(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (a, b):



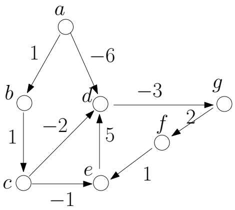
vertex v	$dist(v)$
a	0
b	1
c	∞
d	∞
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (a, d):



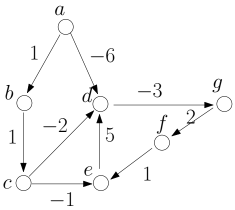
vertex v	$dist(v)$
a	0
b	1
c	∞
d	-6
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (b, c):



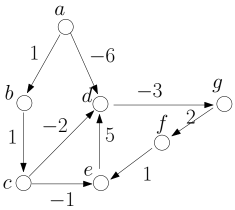
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (c, d):



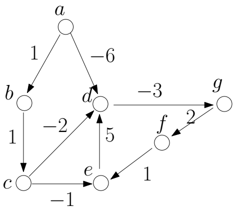
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	∞
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (c, e):



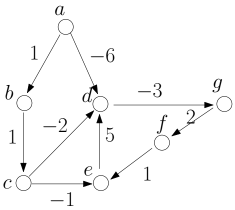
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	∞
g	∞

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (d, g):



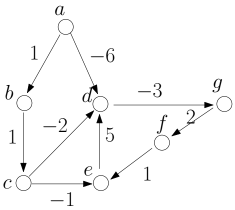
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	∞
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (e, d):



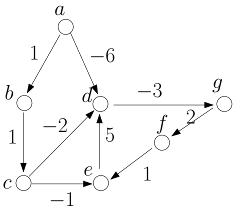
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	∞
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (f, e):



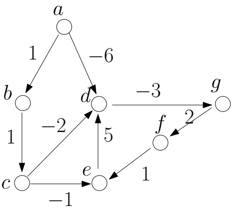
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	∞
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

Relaxing all edges for the **first time**.

Here is what happens after relaxing (g, f):



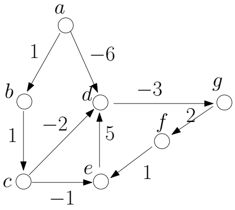
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	1
f	−7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relax all edges for a **second time**.

Here is the content of the table at the end of this relaxation round:



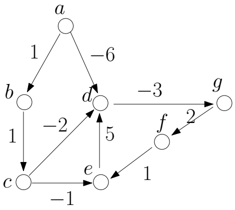
vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	−6
f	−7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relax all edges for a **third time**.

Here is the content of the table at the end of this relaxation round (no changes from the previous round):

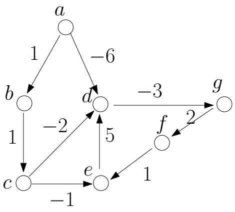


vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	−6
f	−7
g	−9

Alphabetic order of the edges in the graph:
$(a, b), (a, d), (b, c), (c, d), (c, e), (d, g), (e, d), (f, e), (g, f)$.

Example

In the same fashion, relax all edges for a **fourth time**, **fifth time**, and then a **sixth time**. No more changes to the table:



vertex v	$dist(v)$
a	0
b	1
c	2
d	−6
e	−6
f	−7
g	−9

The algorithm then terminates here with the above values as the final shortest path distances.

Remark: We did 6 rounds only to follow the algorithm description faithfully. As a heuristic, we can stop as soon as no changes are made to the table after some round.

Time

The running time is clearly $O(|V||E|)$.

Correctness

Theorem: Consider any vertex v; suppose that there is a shortest path from s to v that has ℓ edges. Then, after ℓ rounds of edge relaxations, it must hold that $dist(v) = spdist(v)$.

Proof:

We will prove the theorem by induction on ℓ. If $\ell = 0$, then $v = s$, in which case the theorem is obviously correct. Next, assuming the statement's correctness for $\ell < i$ where i is an integer at least 1, we will prove it holds for $\ell = i$ as well.

Denote by π the shortest path from s to v, namely, π has i edges. Let p be the vertex right before v on π.

By the inductive assumption, we know that $dist(p)$ was already equal to $spdist(p)$ after the $(i-1)$-th round of edge relaxations.

In the i-th round, by relaxing edge (p, v), we make sure:

$$\begin{aligned} dist(v) &\le dist(p) + w(p, v) \\ &= spdist(p) + w(p, v) \\ &= spdist(v). \end{aligned}$$

□