CSCI3160: Regular Exercise Set 4

Prepared by Yufei Tao

Problem 1. Recall that a *tree* is a connected graph without cycles. Prove:

- Every tree has at least a leaf node, i.e., a node with degree 1 (i.e., a node incident to only one edge).
- Every tree with n nodes has precisely $n - 1$ edges.

Problem 2. Let G be a simple graph with n vertices and $n - 1$ edges. Prove: if G is connected (i.e., a path exists between any two vertices in G), then G must be a tree.

Problem 3 (one for one, still a tree). Let T be a tree. Add a new edge between two vertices in T; this gives us a graph G with a cycle *cyc*. Now, remove from G an arbitrary edge e' of *cyc*; let G' be the graph thus obtained. Prove: G' is a tree.

Problem 4. Let S be a set of integer pairs of the form (id, v). We will refer to the first field as the *id* of the pair, and the second as the *key* of the pair. Design a data structure that supports the following operations:

- Insert: add a new pair (id, v) to S (you can assume that S does not already have a pair with the same id).
- Delete: given an integer t, delete the pair (id, v) from S where $t = id$, if such a pair exists.
- DeleteMin: remove from S the pair with the smallest key, and return it. .

Your structure must consume $O(n)$ space, and support all operations in $O(\log n)$ time where $n = |S|$.

Problem 5. Prove: in a weighted undirected graph $G = (V, E)$ where all the edges have distinct weights, the minimum spanning tree (MST) is unique.

Problem 6. Describe how to implement the Prim's algorithm on a graph $G = (V, E)$ in $O((|V| + |E|) \cdot \log |V|)$ time.