Week 6 Tutorial

CSCI2100 Teaching Team 2021

Department of Computer Science and Engineering

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Pivot Selection

Input: An array A of n integers in arbitrary order.
Output: An element in A whose rank is between $\frac{n}{10}$ and $\frac{9n}{10}$.

Example:

2	3	1	4	5	9	7	6	10	8

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Valid answers: any number from 2 to 9.

Pivot Selection

Algorithm

1. Randomly pick an integer v from A; call v the pivot.
2. Get the rank r of v.
3. If r is not in $[n/10, 9n/10]$, repeat from 1.
4. Otherwise, output v.

Cost Analysis

How many times do we have to repeat Step 1 and 2?

Each run finds a valid answer v with probability $4/5$. Thus, we need to repeat $5/4$ times in expectation.

Hence, our algorithm finishes in $O(n)$ expected time.

> **Think:** If we use the pivot picked in the above manner for k-selection, what is the expected cost of the k-selection algorithm discussed in the lecture?

Pivot Selection

Think: what if

Input: An array A of n integers in arbitrary order.
Output: An element in A whose rank is between $0.4999n$ and $0.5n$?

The next few slides will introduce you to some basic ideas behind generating a random number. As you will see, all we need is the ability to generate a random bit.

Coin Game 1

Given a fair coin, how do you generate a number from 1 to 4 uniformly at random?

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Given a fair coin, how do you generate a number from 1 to 4 uniformly at random?

Solution: Flip the coin twice. Assign numbers as follows:

- (Head, Head): 1
- (Head, Tail): 2
- (Tail, Head): 3
- (Tail, Tail): 4

Coin Game 2

Given a fair coin, how do you generate a number from 1 to 3 uniformly at random?

Hint: Use the previous algorithm as a black box.

Coin Game 1

Given a fair coin, how do you generate a number from 1 to 3 uniformly at random?

Solution: Run the algorithm in Coin Game 1. If the algorithm returns 4, ignore it and run again.

Cost: The number of repeats is $O(1)$.

Coin Game 3

Given a fair coin, how do you generate a number from 1 to n uniformly at random?

Solution: See a regular exercise.

Example: $n = 37$.

1. Generate a number x in $[1, 64]$ uniformly at random.
2. If $x > 37$, repeat step 1.

The number of repeats is $O(1)$.

In the next part of the tutorial, we will show how to sort a **multi-set**.

Sorting a Multi-Set

So far we have assumed the input to sorting is a set S of integers.

What if we want to sort a multi-set A, i.e. a collection of integers which may contain duplicates?

2	3	7	1	4	5	5	6	2	8	6	7

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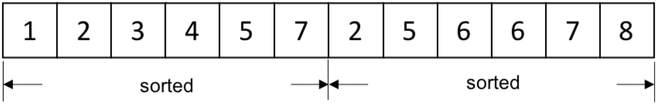
Merge Sort

1. Sort the first half of the array A.
2. Sort the second half of the array A.
3. Consider both subproblems solved and merge the two halves of the array into the final sorted sequence.

We only need to modify Step 3.



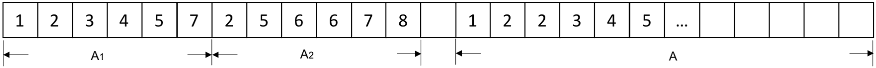
A

Merging

At the beginning, set $i = j = 1$.

Repeat until $i > n/2$ or $j > n/2$:

1. If $A_1[i]$ (i.e., the i-th integer of A_1) is smaller or equal to $A_2[j]$, append $A_1[i]$ to A, and increase $i$by 1.
2. Otherwise, append $A2[j]$ to A, and increase j by 1.



Next, we will show how to break ties using **composite keys**. With this technique, we can turn any comparison-based algorithm designed for sorting **sets** into another algorithm for sorting **multi-sets**.

Composite Keys

1. Convert every integer in A to a key-id pair.
   - E.g. $A[1] \rightarrow (A[1], 1)$.
2. Break tie by comparing the ids.
   - $(a_1, b_1) < (a_2, b_2) \iff a_1 < a_2$ or $a_1 = a_2, b_1 < b_2$.

Example: Convert the array A.

2	3	7	1	4	5	5	6	2	8	6	7

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$(2, 1)$	$(3, 2)$	$(7, 3)$	$(1, 4)$	$(4, 5)$	$(5, 6)$	$(5, 7)$	$(6, 8)$	$(2, 9)$	$(2, 10)$	$(8, 11)$	$(7, 12)$

Bonus: Quick Sort Exercise

Quick Sort Input: An array $A = (5, 9, 3, 10, 26, 37, 14, 12)$.

What is the probability that the algorithm compares the numbers 3 and 37?

Observations:

- Eventually, every integer will be selected as a pivot.
- 3 and 37 are not compared, if any integer between them gets selected as a pivot before 3 and 37.

Example: If 10 is the first pivot, then 3 and 37 will be separated and will not be compared in the rest of the algorithm.

Bonus: Quick Sort Exercise

Solution: 3 and 37 are compared if and only if either one is the first pivot among all integers in A.

The probability is $\frac{2}{|A|} = \frac{1}{4}$.

Bonus: Quick Sort Exercise

Quick Sort Input: An array $A = (5, 9, 3, 10, 26, 37, 14, 12)$.

A more challenging problem:

What is the probability that 3 is compared with 14 in the algorithm?

Solution: 3 and 14 are compared if and only if either one is the first pivot among $3, 5, 9, 10, 12, 14$.

The probability is $\frac{2}{6} = \frac{1}{3}$. (**think:** why?)