CSCI2100: Regular Exercise Set 3

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Problem 1. Prove $\log_2(n!) = \Theta(n \log n)$.

Problem 2. Let $f(n)$ be a function of positive integer n. We know:

$$\begin{aligned} f(1) &= 1 \\ f(n) &\leq 2 + f(\lceil n/10 \rceil). \end{aligned}$$

Prove $f(n) = O(\log n)$. Recall that $\lceil x \rceil$ is the ceiling operator that returns the smallest integer at least x.

Problem 3. Let $f(n)$ be a function of positive integer n. We know:

$$\begin{aligned} f(1) &= 1 \\ f(n) &\leq 2 + f(\lceil 3n/10 \rceil). \end{aligned}$$

Prove $f(n) = O(\log n)$. Recall that $\lceil x \rceil$ is the ceiling operator that returns the smallest integer at least x.

Problem 4. Let $f(n)$ be a function of positive integer n. We know:

$$\begin{aligned} f(1) &= 1 \\ f(n) &\leq 2n + 4f(\lceil n/4 \rceil). \end{aligned}$$

Prove $f(n) = O(n \log n)$.

Problem 5 (Bubble Sort). Let us re-visit the sorting problem. Recall that, in this problem, we are given an array A of n integers, and need to re-arrange them in ascending order. Consider the following *bubble sort* algorithm:

1. If $n = 1$, nothing to sort; return.
2. Otherwise, do the following in ascending order of $i \in [1, n-1]$: if $A[i] > A[i+1]$, swap the integers in $A[i]$ and $A[i+1]$.
3. Recurse in the part of the array from $A[1]$ to $A[n-1]$.

Prove that the algorithm terminates in $O(n^2)$ time.

As an example, support that A contains the sequence of integers $(10, 15, 8, 29, 13)$. After Step 2 has been executed once, array A becomes $(10, 8, 15, 13, 29)$.

Problem 6* (Modified Merge Sort). Let us consider a variant of the merge sort algorithm for sorting an array A of n elements (we will use the notation $A[i..j]$ to represent the part of the array from $A[i]$ to $A[j]$):

- If $n = 1$ then return immediately.
- Otherwise set $k = \lceil n/3 \rceil$.
- Recursively sort $A[1..k]$ and $A[k+1..n]$, respectively.
- Merge $A[1..k]$ and $A[k+1..n]$ into one sorted array.

Prove that this algorithm runs in $O(n \log n)$ time.