CSCI: Regular Exercise Set 2

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Problem 1. Prove $30\sqrt{n} = O(\sqrt{n})$.

Solution. Set $c_1 = 30$ and $c_2 = 1$. The inequality $30\sqrt{n} \le c_1\sqrt{n}$ holds for all $n \ge c_2$. This completes the proof.

Problem 2. Prove $\sqrt{n} = O(n)$.

Solution. Set $c_1 = 1$ and $c_2 = 1$. The inequality $\sqrt{n} \le c_1 n$ holds for all $n \ge c_2$. This completes the proof.

Problem 3. Let $f(n)$, $g(n)$, and $h(n)$ be functions of integer n. Prove: if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Solution. Since $f(n) = O(g(n))$, there exist constants c_1, c_2 such that for all $n \ge c_2$, it holds that

$$f(n) \quad \le \quad c_1 g(n).$$

Similarly, since $g(n) = O(h(n))$, there exist constants c_1', c_2' such that for all $n \ge c_2'$, it holds that

$$g(n) \quad \le \quad c_1' h(n).$$

Set $c_1'' = c_1 c_1'$ and $c_2'' = \max\{c_2, c_2'\}$. From the above, we know that for all $n \ge c_2''$, it holds that

$$f(n) \quad \le \quad c_1 g(n) \le c_1 c_1' h(n) = c_1'' h(n).$$

Therefore, $f(n) = O(h(n))$.

Problem 4. Prove $(2n + 2)^3 = O(n^3)$.

Solution. Set $c_1 = 4^3$ and $c_2 = 1$. The inequality $(2n + 2)^3 \le c_1 n^3$ holds for all $n \ge c_2$. This completes the proof.

Problem 5. Prove or disprove: $4^n = O(2^n)$.

Solution. Consider the ratio $4^n/2^n$, which equals 2^n. The ratio clearly goes to ∞ when n tends to ∞. Therefore, the statement is incorrect.

Problem 6. Prove or disprove: $\frac{1}{n} = O(1)$.

Solution. Set $c_1 = 1$ and $c_2 = 1$. The inequality $1/n \le c_1 \cdot 1$ holds for all $n \ge c_2$. This completes the proof.

Problem 7*. Prove that if $k \log_2 k = \Theta(n)$, then $k = \Theta(n/\log n)$.

Solution. Since $k \log_2 k = O(n)$, there exist constants c_1, c_2 such that $k \log_2 k \le c_1 n$ for all $n \ge c_2$. On the other hand, $k \log_2 k = \Omega(n)$ indicates the existence of constants c_1', c_2' such that $k \log_2 k \ge c_1' n$ for all $n \ge c_2'$. Therefore, for all $n \ge \max\{c_2, c_2'\}$, we have:

$$c_1' n \le k \log_2 k \le c_1 n. \tag{1}$$

Set $c_2'' = \max\{c_1, c_2, c_2'\}$.

When $n \geq \max\{c_2'', (1/c_1')^2\}$, we derive from (1):

$$\begin{aligned}
&\log_2(c_1' n) \leq \log_2(k \log_2 k) \leq \log_2(c_1 n) \\
&\Rightarrow \log_2 c_1' + \log_2 n \leq \log_2 k + \log_2 \log_2 k \leq \log_2 c_1 + \log_2 n \\
&\Rightarrow \begin{cases} \log_2 k \leq \log_2 c_1 + \log_2 n \leq 2 \log_2 n & \text{(using } n \geq c_1) \\ 2 \log_2 k \geq \log_2 k + \log_2 \log_2 k \geq \log_2 c_1' + \log_2 n \geq \frac{1}{2} \log_2 n \end{cases} \\
&\Rightarrow \frac{\log_2 n}{4} \leq \log_2 k \leq 2 \log_2 n.
\end{aligned} \tag{2}$$

Combining (1) and (2) leads to

$$\begin{cases} k \leq c_1 \frac{n}{\log_2 k} \leq 4c_1 \frac{n}{\log_2 n} \\ k \geq c_1' \frac{n}{\log_2 k} \geq \frac{c_1'}{2} \frac{n}{\log_2 n} \end{cases}$$

which means $k = \Theta(n / \log n)$.

Problem 8. We can extend the big-O notation to multiple variables. In this problem, we will focus on two variables, but the idea extends to more variables in a straightforward manner.

Formally, let $f(n, m)$ and $g(n, m)$ be functions of variables n and m satisfying $f(n, m) \geq 0$ and $g(n, m) \geq 0$. We say $f(n, m) = O(g(n, m))$ if there exist constants c_1 and c_2 such that $f(n, m) \leq c_1 \cdot g(n, m)$ holds for all $n \geq c_2$ and $m \geq c_2$.

Prove:

- $n^2 m + 100nm = O(n^2 m)$.
- $n^2 m + 100nm^2 = O(n^2 m + nm^2)$.

Solution. There exist constants $c_1 = 101$ and $c_2 = 1$ such that $n^2 m + 100nm \leq c_1 n^2 m$ holds for all $n \geq c_2$ and $m \geq c_2$. Therefore, $n^2 m + 100nm = O(n^2 m)$.

There exist constants $c_1 = 100$ and $c_2 = 1$ such that $n^2 m + 100nm \leq c_1(n^2 m + nm^2)$ holds for all $n \geq c_2$ and $m \geq c_2$. Therefore, $n^2 m + 100nm = O(n^2 m + nm^2)$.