ENGG1410-F Tutorial 7

Hao Xu

Department of Computer Science and Engineering
The Chinese University of Hong Kong

Problem 1. Orthogonal set

Consider the following set $\boldsymbol{S}$ with three column vectors:

$$\boldsymbol{S} = \left\{ \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes $\boldsymbol{S}$ an **orthogonal set**.

Solution

For S to be orthogonal, the vectors in S must be mutually orthogonal to each other. We therefore have:

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

which gives the following set of equations on variables x, y, and z:

$$-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}z = 0$$

$$y = 0$$

Solution–cont.

The set of solutions $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is:

$$\boldsymbol{S} = \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} |t \in \mathbb{R} \right\}$$

Problem 2. Orthogonal matrix

Consider the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0 & x \\ 0 & 1 & y \\ \frac{\sqrt{2}}{2} & 0 & z \end{bmatrix}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes $\boldsymbol{A}$ an **orthogonal matrix**.

Recall that matrix $\boldsymbol{A}$ is orthogonal if and only if both conditions below are satisfied:

- All column vectors are mutually orthogonal.
- All column vectors have unit length.

Solution

In Problem1, we have already obtained the set of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the first bullet(the orthogonal constraint):

$$\boldsymbol{S} = \left\{ \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} | t \in \mathbb{R} \right\}$$

To satisfy the "unit length" constraint, we need:

$$\sqrt{x^2 + y^2 + z^2} = 1 \Rightarrow$$

$$t^2 + t^2 = 1 \Rightarrow$$

$$t = \frac{\sqrt{2}}{2} \; or \; -\frac{\sqrt{2}}{2}$$

Hence, there are only two $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that can make A orthogonal:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Problem 3. Symmetric Matrix Diagonalization

Diagonalize the following symmetric matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

into $\boldsymbol{QBQ^{-1}}$ where $\boldsymbol{B}$ is a diagonal matrix, and $\boldsymbol{Q}$ is an **orthogonal matrix**. You only need to give the details of $\boldsymbol{Q}$ and $\boldsymbol{B}$.

Hint: $\boldsymbol{A}$ has only two eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = 3$.

Solution

We aim to obtain three eigenvectors of $\boldsymbol{A}$, denoted as $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ respectively, that are (i) mutually orthogonal to each other and (ii) have lengths 1.

We first calculate the eigenspace of λ_1:

$$EigenSpace(\lambda_1) = \left\{ \begin{bmatrix} u \\ v \\ -u-v \end{bmatrix} | u, v \in \mathbb{R} \right\}$$

$EigenSpace(\lambda_1)$ has dimension 2. We will first take from the set two eigenvectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ that are orthogonal to each other.

But how?

Solution–cont.

We first set $\boldsymbol{x}_1$ to an arbitrary non-zero vector, e.g., $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Regarding $\boldsymbol{x}_2 = \begin{bmatrix} u \\ v \\ -u-v \end{bmatrix}$, we ensure orthogonality between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ by requiring their dot product to be 0:

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ -u-v \end{bmatrix} &= 0 \\ u + u + v &= 0 \\ 2u + v &= 0 \end{aligned}$$

Any non-zero vector satisfying the above equation and in the form of $\begin{bmatrix} u \\ v \\ -u-v \end{bmatrix}$ will be perpendicular to $\boldsymbol{x}_1$.

Solution–cont.

We can set u to any value such that $\boldsymbol{x}_2$ is not a zero-vector, e.g., $u = 1$ which gives $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Finally, normalize $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to have length 1, which gives

$\boldsymbol{v}_1 = \frac{\boldsymbol{x}_1}{|\boldsymbol{x}_1|} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ and

$\boldsymbol{v}_2 = \frac{\boldsymbol{x}_2}{|\boldsymbol{x}_2|} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.

Solution–cont.

Next, take eigenvector from $EigenSpace(\lambda_2)$:

$$EigenSpace(\lambda_2) = \left\{ \begin{bmatrix} u \\ u \\ u \end{bmatrix} | u \in \mathbb{R} \right\}$$

This set has dimension 1. We take an arbitrary eigenvector, e.g., $\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and normalizing this vector to length 1 gives $\boldsymbol{v}_3 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

Therefore:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 4.

Suppose that an $n \times n$ matrix $\boldsymbol{A}$ can be computed as $\boldsymbol{QBQ^{-1}}$ where $\boldsymbol{Q}$ is an $n \times n$ orthogonal matrix, and $\boldsymbol{B}$ is an $n \times n$ diagonal matrix.
Prove: $\boldsymbol{A}$ is a symmetric matrix.