Lecture Notes: Determinant of a Square Matrix

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1 Determinant Definition

Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix (i.e., $\boldsymbol{A}$ is a square matrix). Given a pair of (i, j), we define $\boldsymbol{A}_{ij}$ to be the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and j-th column of $\boldsymbol{A}$. For example, suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix}$$

Then:

$$\boldsymbol{A}_{21} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \boldsymbol{A}_{22} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \boldsymbol{A}_{32} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

We are now ready to define determinants:

Definition 1. *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. If $n = 1$, its* **determinant**, *denoted as $det(\boldsymbol{A})$, equals a_{11}. If $n > 1$, we first choose an arbitrary $i^* \in [1, n]$, and then define the determinant of $\boldsymbol{A}$ recursively as:*

$$det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{i^*+j} \cdot a_{i^*j} \cdot det(\boldsymbol{A}_{i^*j}). \tag{1}$$

Besides $det(\boldsymbol{A})$, we may also denote the determinant of $\boldsymbol{A}$ as $|\boldsymbol{A}|$. Henceforth, if we apply (1) to compute $det(\boldsymbol{A})$, we say that we *expand $\boldsymbol{A}$ by row i^**. It is important to note that the value of $det(\boldsymbol{A})$ does *not* depend on the choice of i^*. We omit the proof of this fact, but illustrate it in the following examples.

Example 1 (Second-Order Determinants). In general, if $\boldsymbol{A} = [a_{ij}]$ is a 2×2 matrix, then

$$det(\boldsymbol{A}) = a_{11}a_{22} - a_{12}a_{21}.$$

For instance:

$$\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 2 \times 2 - 1 \times (-1) = 5.$$

We may verify the above by definition as follows. Choosing $i^* = 1$, we get:

$$\begin{aligned} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} &= (-1)^{1+1} \cdot 2 \cdot det(\boldsymbol{A}_{11}) + (-1)^{1+2} \cdot 1 \cdot det(\boldsymbol{A}_{12}) \\ &= 2 \times 2 + (-1) \times (-1) = 5. \end{aligned}$$

Alternatively, choosing $i^* = 2$, we get:

$$\begin{vmatrix} 2 & 1 \\ \text{-1} & 2 \end{vmatrix} = (-1)^{2+1} \cdot (-1) \cdot det(\boldsymbol{A}_{21}) + (-1)^{2+2} \cdot 2 \cdot det(\boldsymbol{A}_{22})$$
$$= 1 \times 1 + 2 \times 2 = 5.$$

□

Example 2 (Third-Order Determinants). Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & \text{-2} \\ \text{-1} & \text{-1} & 2 \end{bmatrix}$$

Choosing $i^* = 1$, we get:

$$det(\boldsymbol{A}) = 1\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ -1 & -1 \end{vmatrix}$$
$$= 1(0-2) - 2(6-2) + 1(-3-0) = -13.$$

Alternatively, choosing $i^* = 2$, we get:

$$det(\boldsymbol{A}) = -3\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix}$$
$$= (-3)(4+1) + 0(2+1) + 2(-1+2) = -13.$$

□

2 Properties of Determinants

Expansion by a Column. Definition 1 allows us to compute the determinant of a matrix by row expansion. We may also achieve the same purpose by column expansion.

Lemma 1. *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix with $n > 1$. Choose an arbitrary $j^* \in [1, n]$. The determinant of $\boldsymbol{A}$ equals:*

$$det(\boldsymbol{A}) = \sum_{i=1}^{n} (-1)^{i+j^*} \cdot a_{ij^*} \cdot det(\boldsymbol{A}_{ij^*}).$$

The value of the above equation does not depend on the choice of j^*.

We omit the proof but illustrate the lemma with an example below. Henceforth, if we compute $det(\boldsymbol{A})$ by the above lemma, we say that we *expand $\boldsymbol{A}$ by column j^*.*

Example 3. Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & \text{-2} \\ \text{-1} & \text{-1} & 2 \end{bmatrix}$$

Choosing $j^* = 1$, we get:

$$\begin{aligned} det(\boldsymbol{A}) &= 1\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} \\ &= 1(0-2) - 3(4+1) - 1(-4-0) = -13. \end{aligned}$$

□

Corollary 1. *Let $\boldsymbol{A}$ be a square matrix. Then, $det(\boldsymbol{A}) = det(\boldsymbol{A}^T)$.*

Proof. Note that expanding $\boldsymbol{A}$ by column k is equivalent to expanding $\boldsymbol{A}^T$ by row k. □

Corollary 2. *If $\boldsymbol{A}$ has a zero row or a zero column, then $det(\boldsymbol{A}) = 0$.*

Proof. If $\boldsymbol{A}$ has a zero row, then $det(\boldsymbol{A}) = 0$ follows from expanding $\boldsymbol{A}$ by that row. The case where $\boldsymbol{A}$ has a zero column is similar. □

Determinant of a Row-Echelon Matrix. The next lemma shows that the determinant of a matrix in row-echelon form is simply the product of the values on the main diagonal.

Lemma 2. *Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix in row-echelon form. Then, $det(\boldsymbol{A}) = \Pi_{i=1}^{n} a_{ii}$.*

Proof. We can prove the lemma by induction. First, correctness is obvious for $n = 1$. Assuming correctness for $n \leq t - 1$ (for $t \geq 2$), consider $n = t$. Expanding $\boldsymbol{A}$ by the first row gives:

$$det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{1+j} \cdot a_{1j} \cdot det(\boldsymbol{A}_{1j}). \quad (2)$$

From induction we know that $det(\boldsymbol{A}_{11}) = \Pi_{i=2}^{n} a_{ii}$. Furthermore, for $j > 1$, $det(\boldsymbol{A}_{1j}) = 0$ because the first column of $\boldsymbol{A}_{1j}$ contains all 0's. It thus follows that (2) equals $\Pi_{i=1}^{n} a_{ii}$. □

Determinants under Elementary Row Operations. Let $\boldsymbol{A} = [a_{ij}]$ be an $n \times n$ matrix. Recalled that the elementary row operations on $\boldsymbol{A}$ are:

1. Switch two rows of $\boldsymbol{A}$.
2. Multiply all numbers of a row of $\boldsymbol{A}$ by the same non-zero value c.
3. Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ be two distinct row vectors of $\boldsymbol{A}$. Update row $\boldsymbol{r}_i$ to $\boldsymbol{r}_i + \boldsymbol{r}_j$.

Next, we refer to the above as Operation 1, 2, and 3, respectively.

Lemma 3. *The determinant of $\boldsymbol{A}$*

1. *should be multiplied by -1 after Operation 1;*
2. *should be multiplied by c after Operation 2;*
3. *has no change after Operation 3.*

Proof. See appendix. □

The following corollary will be very useful:

Corollary 3. *Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ be two distinct row vectors of $\boldsymbol{A}$. The determinant of $\boldsymbol{A}$ does not change after the following operation:*

- *Update row $\boldsymbol{r}_i$ to $\boldsymbol{r}_i + c \cdot \boldsymbol{r}_j$, where c is a real value.*

Proof. We consider only $c \neq 0$ (the case of $c = 0$ is trivial). Let $\boldsymbol{A'}$ be the array after applying the above operation. We can also obtain $\boldsymbol{A'}$ by performing the next three operations:

1. Multiply the j-th row of $\boldsymbol{A}$ by c. Let $\boldsymbol{A_1}$ be the array obtained.
2. Add the j-th row of $\boldsymbol{A_1}$ into its i-th row. Let $\boldsymbol{A_2}$ be the array obtained.
3. Multiply the j-th row of $\boldsymbol{A_2}$ by $1/c$. Let $\boldsymbol{A_3}$ be the array obtained. Note that $\boldsymbol{A_3} = \boldsymbol{A'}$.

By Lemma 3, $det(\boldsymbol{A_1}) = c \cdot det(\boldsymbol{A})$, $det(\boldsymbol{A_2}) = det(\boldsymbol{A_1})$, and $det(\boldsymbol{A_3}) = (1/c) \cdot det(\boldsymbol{A_3})$. Hence, $det(\boldsymbol{A}) = det(\boldsymbol{A'})$. □

Let us illustrate Lemma 3 and Corollary 3 with an example.

Example 4.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & -6 & -5 \\ 0 & 0 & 13/6 \end{vmatrix} = -13.$$

Here is another derivation giving the same result:

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ -1 & -1 & 2 \\ 3 & 0 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -6 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 13 \end{vmatrix} = -13.$$

□

Corollary 4. *If $\boldsymbol{A}$ has two identical rows or columns, then $det(\boldsymbol{A}) = 0$.*

Proof. We prove only the row case. Switching the two rows gets back the same matrix. However, by Lemma 3, the determinant of the matrix should be multiplied by -1. Therefore, we get $det(\boldsymbol{A}) = -det(\boldsymbol{A})$, meaning $det(\boldsymbol{A}) = 0$. □

Determinant under Matrix Multiplication. The following is a perhaps surprising property of determinants:

Lemma 4. *Let $\boldsymbol{A}, \boldsymbol{B}$ be $n \times n$ matrices. It holds that $det(\boldsymbol{AB}) = det(\boldsymbol{A}) \cdot det(\boldsymbol{B})$.*

The proof is not required, but we will discuss it in a tutorial after we have learned the concept of "matrix inversion".

Example 5.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{vmatrix} = -13$$

$$\begin{vmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = -3$$

$$\left| \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -2 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \right| = \begin{vmatrix} -1 & 1 & 2 \\ -8 & 7 & 0 \\ 4 & -6 & -1 \end{vmatrix} = 39.$$

□

Relationships with Ranks. The lemma below relates determinants to ranks:

Lemma 5. *Let* $\boldsymbol{A}$ *be an* $n \times n$ *matrix.* $\boldsymbol{A}$ *has rank* n *if and only if* $det(\boldsymbol{A}) \neq 0$.

Proof. We can first apply elementary row operations to convert $\boldsymbol{A}$ into row-echelon form $\boldsymbol{A}^*$. Thus, $\boldsymbol{A}$ has rank n if and only if $\boldsymbol{A}^*$ has rank n. Since $\boldsymbol{A}^*$ is a square matrix, that it has rank n is equivalent to saying that all the numbers on its main diagonal are non-zero. Thus, by Lemma 2, we know that $\boldsymbol{A}^*$ has rank n if and only if $det(\boldsymbol{A}^*) \neq 0$. Finally, by Lemma 3, $det(\boldsymbol{A}) \neq 0$ if and only if $det(\boldsymbol{A}^*) \neq 0$. We thus complete the proof. □

Appendix: Proof of Lemma 3

The claim on Operation 2 is easy to prove; we leave the proof to you. Regarding the other two operations, we will first prove the claim on Operation 3, and then the claim on Operation 1.

Proof of the Claim on Operation 3. Let us revisit the claim of Corollary 4, restated below:

Fact 1: If $\boldsymbol{A}$ has two identical rows or columns, then $det(\boldsymbol{A}) = 0$.

The proof of Corollary 4 was based on Lemma 3, and hence, cannot be used here because we are actually *proving* Lemma 3. Next, we give an alternative argument that establishes Fact 1 directly, without using Lemma 3.

Proof of Fact 1. If $\boldsymbol{A}$ is a 2×2 matrix, the fact can be easily verified. Inductively, assuming that the fact holds for any $(n-1) \times (n-1)$ matrix (for $n \geq 3$), next we prove it for an $n \times n$ matrix $\boldsymbol{A}$ as well.

Without loss of generality, suppose that the a-th and b-th rows of $\boldsymbol{A}$ are identical. Let i be an arbitrary integer in $[1, n]$ such that $i \neq a$ and $i \neq b$; note that i definitely exists because $\boldsymbol{A}$ has at least 3 rows. Let us calculate $det(\boldsymbol{A})$ by expanding $\boldsymbol{A}$ on row i:

$$det(\boldsymbol{A}) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot det(\boldsymbol{A}_{ij}) \tag{3}$$

where a_{ij} is the element of $\boldsymbol{A}$ at the i-th row and the j-th column, and $\boldsymbol{A}_{ij}$ is the submatrix of $\boldsymbol{A}$ after removing the i-th row and the j-th column. The crucial observation is that $\boldsymbol{A}_{ij}$ has two identical rows (i.e., the rows corresponding to "row a" and "row b" of $\boldsymbol{A}$), and hence, $det(\boldsymbol{A}_{ij}) = 0$ by the inductive assumption. This means that (3) must be equivalent to 0. □

We now proceed to prove the claim on Operation 3, levering Fact 1. Suppose that, after performing Operation 3 on $\boldsymbol{A}$, we obtain a matrix $\boldsymbol{A'}$. Our goal is to show that $det(\boldsymbol{A}) = det(\boldsymbol{A'})$. Let us define a new matrix $\boldsymbol{B}$:

- $\boldsymbol{B}$ is the same as $\boldsymbol{A}$, except that the i-th row of $\boldsymbol{B}$ is replaced by the j-th row of $\boldsymbol{A}$.

In other words, the i-th row of $\boldsymbol{B}$ is identical to the j-th row of $\boldsymbol{B}$. Corollary 4 tells us that $det(\boldsymbol{B}) = 0$. Next, we will focus on showing:

$$det(\boldsymbol{A'}) \quad = \quad det(\boldsymbol{A}) + det(\boldsymbol{B}) \tag{4}$$

which will indicate $det(\boldsymbol{A}) = det(\boldsymbol{A'})$ and hence will complete the proof.

Define a'_{ik} as the number at the i-th row and j-th column of $\boldsymbol{A'}$, and define a_{ik}, b_{ik} similarly with respect to $\boldsymbol{A}$, $\boldsymbol{B}$, respectively. Note that:

$$a'_{ik} \quad = \quad a_{ik} + b_{ik}$$

holds by the way $\boldsymbol{A'}$ and $\boldsymbol{B}$ were obtained.

In fact, (4) follows almost directly from the definition of determinants. Let us calculate $det(\boldsymbol{A'})$ by expanding the matrix on row i:

$$\begin{aligned}
det(\boldsymbol{A'}) &= \sum_{k=1}^{n} (-1)^{i+k} a'_{ik} \cdot det(\boldsymbol{A'}_{ik}) \\
&= \sum_{k=1}^{n} (-1)^{i+k} (a_{ik} + b_{ik}) \cdot det(\boldsymbol{A'}_{ik}) \\
&= \left(\sum_{k=1}^{n} (-1)^{i+k} a_{ik} \cdot det(\boldsymbol{A'}_{ik}) \right) + \left(\sum_{k=1}^{n} (-1)^{i+k} b_{ik} \cdot det(\boldsymbol{A'}_{ik}) \right) \\
&= \left(\sum_{k=1}^{n} (-1)^{i+k} a_{ik} \cdot det(\boldsymbol{A}_{ik}) \right) + \left(\sum_{k=1}^{n} (-1)^{i+k} b_{ik} \cdot det(\boldsymbol{B}_{ik}) \right) \\
&= det(\boldsymbol{A}) + det(\boldsymbol{B}).
\end{aligned}$$

Proof of the Claim on Operation 1. Denote the row vectors of $\boldsymbol{A}$ as $\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_n$ respectively. Suppose that Operation 1 switches row i with row j. Denote by $\boldsymbol{B}$ the matrix obtained after the operation. We have:

$$
\begin{aligned}
det(\boldsymbol{A}) &= \begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ \boldsymbol{r}_i \\ ... \\ \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_n \end{vmatrix} = \begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ \boldsymbol{r}_i + \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_n \end{vmatrix} \quad \text{(by Operation 3)} \\
&= -\begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ -\boldsymbol{r}_i - \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_n \end{vmatrix} \quad \text{(by Operation 2)} \\
&= -\begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ -\boldsymbol{r}_i - \boldsymbol{r}_j \\ ... \\ -\boldsymbol{r}_i \\ ... \\ \boldsymbol{r}_n \end{vmatrix} \quad \text{(by Operation 3)} \\
&= \begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ -\boldsymbol{r}_i - \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_i \\ ... \\ \boldsymbol{r}_n \end{vmatrix} \quad \text{(by Operation 2)} \\
&= \begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ -\boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_i \\ ... \\ \boldsymbol{r}_n \end{vmatrix} \quad \text{(by Operation 3)} \\
&= -\begin{vmatrix} \boldsymbol{r}_1 \\ ... \\ \boldsymbol{r}_j \\ ... \\ \boldsymbol{r}_i \\ ... \\ \boldsymbol{r}_n \end{vmatrix} \quad \text{(by Operation 2)} = -det(\boldsymbol{B}).
\end{aligned}
$$

This completes the proof.