Exercises: Dimensions, Spans, and Linear Transformations

In the following exercises, $\mathbb{R}$ denotes the set of all real numbers.

Problem 1. Let V be the set of following 1×4 vectors:

$$[3, 0, 1, 2]$$
$$[6, 1, 0, 0]$$
$$[12, 1, 2, 4]$$
$$[6, 0, 2, 4]$$
$$[9, 0, 1, 2]$$

Find the dimension of V.

Solution. Since the matrix

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 6 & 0 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{bmatrix}$$

has rank 2 (see the exercise list on “Matrix Rank”), the dimension of V is 2.

Problem 2. Let V be the set of 1×4 vectors $[2x - 3y, x + 2y, -y, 4x]$ with $x, y \in \mathbb{R}$. Find the dimension of V and give a basis of V.

Solution. Denote by V' the set of 1×2 vectors $[x, y]$ with $x, y \in \mathbb{R}$. V is obtained from V' through a linear transformation. Clearly the dimension of V' is 2 (here is a basis for V': $\{[1, 0], [0, 1]\}$). Thus, the dimension of V is at most 2. To prove that the dimension of V is exactly 2, it suffices to find two vectors in V that are linearly independent. The following are two such vectors: $[2, 1, 0, 4]$ (given by $x = 1, y = 0$) and $[-3, 2, -1, 0]$ (given by $x = 0, y = 1$). They also form a basis of V.

Problem 3. For each set V of vectors given below, find its dimension and give a basis:

- (a) V is the set of 2D points given by $y = x$ (here, we regard each point (x, y) as a 1×2 vector $[x, y]$);
- (b) V is the set of 2D points given by $y = x + 1$.

Solution. (a) Dimension 1. A basis: $\{[1, 1]\}$.
(b) Dimension 2. A basis: $\{[0, 1], [-1, 0]\}$.

Problem 4. Let V_1 be the set of vectors $[x_1, x_2]^T$ where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$. Define:

$$\begin{aligned} y_1 &= 3x_1 + 2x_2 \\ y_2 &= 4x_1 + x_2 \end{aligned}$$

Let V_2 be the set of vectors $[y_1, y_2]^T$ obtained by applying the above to all vectors $[x_1, x_2]^T \in V_1$. Answer the following questions:

(a) Give the matrix $\boldsymbol{A}$ in the linear transformation $[y_1, y_2]^T = \boldsymbol{A}[x_1, x_2]^T$ from V_1 to V_2.

(b) It is known that there is a linear transformation $[x_1, x_2]^T = \boldsymbol{A'}[y_1, y_2]^T$ from V_2 to V_1. Give the details of the matrix $\boldsymbol{A'}$.

Solution. (a) The transformation can be written as:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) The matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ has rank 2. Hence, it has an inverse $\boldsymbol{A}^{-1}$. Observe that:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

leads to

$$\boldsymbol{A}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

By applying Gauss-Jordan elimination, we can get $\boldsymbol{A}^{-1} = \begin{bmatrix} -1/5 & 2/5 \\ 4/5 & -3/5 \end{bmatrix}$. Therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/5 & 2/5 \\ 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Problem 5. Let V be a set of $1 \times n$ vectors. Let V' be the *projection* of V on the first $t < n$ components, namely:

$$V' = \Big\{[x_1, x_2, ..., x_t] \mid [x_1, x_2, ..., x_t, x_{t+1}, ..., x_n] \in V\Big\}.$$

Prove: the dimension of V is at least the dimension of V'.

For example, if V is the set of 5 vectors in Problem 1 and $t = 2$, then V' is the set of following vectors:

$$[3, 0]$$
$$[6, 1]$$
$$[12, 1]$$
$$[6, 0]$$
$$[9, 0].$$

Solution. For a row vector $\boldsymbol{v}$, we will denote by $\boldsymbol{v}[i]$ the i-th element of $\boldsymbol{v}$. Let d' be the dimension of V'. This means that we can find d' $1 \times t$ vectors $\boldsymbol{v'}_1, ..., \boldsymbol{v'}_t$ in V' that are linearly independent. Remember that each $\boldsymbol{v'}_i$ must come from a vector $\boldsymbol{v}_i \in V$, for $1 \le i \le t$. The vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_{d'}$ must be linearly independent. Otherwise, suppose

$$c_1 \cdot \boldsymbol{v}_1 + ... + c_{d'} \cdot \boldsymbol{v}_{d'} = 0$$

for some real numbers $c_1, ..., c_{d'}$ that are not all 0. Then it must hold that

$$c_1' \cdot \boldsymbol{v'}_1 + ... + c_{d'} \cdot \boldsymbol{v'}_{d'} = 0$$

contradicting the fact that $\boldsymbol{v'}_1, ..., \boldsymbol{v'}_t$ are linearly independent.

Problem 6 (Hard). Consider the following system of linear equations:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let V be the set of 5×1 vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ that satisfy the equation. Prove that V has dimension 2, and find a basis of V.

Solution. The system can be transformed into:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that we can derive all the solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ as follows. First, set x_4, x_5 to any real numbers (i.e., they are unconstrained). Then, solve x_1, x_2, x_3 as:

$$\begin{aligned} x_1 &= -(x_4 + x_5) \\ x_2 &= -x_5 \\ x_3 &= -x_5. \end{aligned} \tag{1}$$

Denote by V' the set of all vectors $\begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$. It is clear that V' has dimension 2 (remember: x_4, x_5 are *unconstrained*). V can be obtained from V' through a linear transformation. Therefore, the dimension of V is at most the dimension of V'. In other words, the dimension of V is at most 2.

On the other hand, note that V' is the projection of V onto the 4-th and 5-th components. From the result of Problem 4, we know that the dimension of V is at least the dimension of V'. In other words, the dimension of V is at least 2.

We now conclude that the dimension of V is precisely 2.

To find a basis of V, simply set $\begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. The former gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and the latter gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 7 (Hard). Consider the following linear system about $\boldsymbol{x}$

$$\boldsymbol{A}\boldsymbol{x} \quad = \quad \mathbf{0}$$

where $\boldsymbol{A}$ is an $m \times n$ coefficient matrix, and $\boldsymbol{x}$ an $n \times 1$ matrix. Let V be the set of all such $\boldsymbol{x}$ satisfying the system. Suppose that the rank of $\boldsymbol{A}$ is $r < n$. Prove that V has dimension $n - r$.

Solution. Let $\boldsymbol{B}$ be a row echelon form of $\boldsymbol{A}$. We know that $\boldsymbol{B}$ has exactly r non-zero rows.

The solutions $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}$ of the system can be obtained as follows. First, fix $\begin{bmatrix} x_{r+1} \\ x_{r+2} \\ ... \\ x_n \end{bmatrix}$ to an arbitrary $(n - r) \times 1$ vector. Then, the r non-zero rows of $\boldsymbol{B}$ give a linear system with respect to $x_1, x_2, ..., x_r$ (treating $x_{r+1}, x_{r+2}, ..., x_r$ as constants). This linear system has a unique solution.

Therefore, V is the set of all outputs of a linear function $\boldsymbol{f}(x_{r+1}, x_{r+2}, ..., x_n)$ where (i) each output of $\boldsymbol{f}$ is an n-dimensional vector $\boldsymbol{v}$, and (ii) $x_{r+1}, x_{r+2}, ..., x_n$ can be arbitrary real values. In other words, $\boldsymbol{f}$ is in fact a linear transformation from the set of all possible $(n - r) \times 1$ vectors to V. It thus follows that the dimension of V is at most $n - r$.

On the other hand, since the projection of V onto the components $x_{r+1}, ..., x_n$ is the set of all possible $(n - r) \times 1$ vectors. It follows from the result of Problem 4 that V has dimension at least $n - r$.

We now conclude that the dimension of V is exactly $n - r$. $\square$