Exercises: Eigenvalues and Eigenvectors

Problem 1. Find all the eigenvalues and eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution. Let λ be an eigenvalue of $\boldsymbol{A}$. To obtain all possible λ, we solve the characteristic equation of $\boldsymbol{A}$ (let $\boldsymbol{I}$ be the 3×3 identity matrix):

$$\begin{aligned} det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= 0 \Rightarrow \\ \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} &= 0 \Rightarrow \\ (\lambda - 1)^2(\lambda + 1) &= 0 \end{aligned}$$

Hence, $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$.

To find all the eigenvectors of $\lambda_1 = 1$, we need to solve $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ from:

$$\begin{aligned} (\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

The set of solutions to the above equation—*EigenSpace*(λ_1)—includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= v \\ x_3 &= u \end{aligned}$$

for any $u, v \in \mathbb{R}$. Any non-zero vector in *EigenSpace*(λ_1) is an eigenvector of $\boldsymbol{A}$ corresponding to λ_1.

Similarly, to find all the eigenvectors of $\lambda_2 = -1$, we need to solve $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ from:

$$\begin{aligned} (\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

The set of solutions to the above equation—*EigenSpace*(λ_2)—includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= 0 \\ x_3 &= -u \end{aligned}$$

for any $u \in \mathbb{R}$. Any non-zero vector in $EigenSpace(\lambda_2)$ is an eigenvector of $\boldsymbol{A}$ corresponding to λ_2.

Problem 2. Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Prove: $\boldsymbol{A}$ and $\boldsymbol{A}^T$ have exactly the same eigenvalues.

Proof. Recall that an eigenvalue of a matrix is a root of the matrix's characteristic equation, which equates the matrix's characteristic polynomial to 0. It suffices to show that the characteristic polynomial of $\boldsymbol{A}$ is the same as that of $\boldsymbol{A}^T$. In other words, we want to show that $det(\boldsymbol{A} - \lambda \boldsymbol{I}) = det(\boldsymbol{A}^T - \lambda \boldsymbol{I})$. This is true because $\boldsymbol{A} - \lambda \boldsymbol{I} = (\boldsymbol{A}^T - \lambda \boldsymbol{I})^T$. □

Problem 3 (Hard). Let $\boldsymbol{A}$ be an $n \times n$ square matrix. Prove: $\boldsymbol{A}^{-1}$ exists if and only if 0 is not an eigenvalue of $\boldsymbol{A}$.

Proof. *If-Direction.* The objective is to show that if 0 is not an eigenvalue of $\boldsymbol{A}$, then $\boldsymbol{A}^{-1}$ exists, namely, the rank of $\boldsymbol{A}$ is n. Suppose, on the contrary, that the rank of $\boldsymbol{A}$ is less than n. Consider the linear system $\boldsymbol{Ax} = \boldsymbol{0}$ where $\boldsymbol{x}$ is an $n \times 1$ matrix. The hypothesis that $rank\, \boldsymbol{A} < n$ indicates that the system has infinitely many solutions. In other words, there exists a non-zero $\boldsymbol{x}$ satisfying $\boldsymbol{Ax} = \boldsymbol{0x} = \boldsymbol{0}$. This, however, indicates that 0 is an eigenvalue of $\boldsymbol{A}$, which is a contradiction.

Only-If Direction. The objective is to show that if $\boldsymbol{A}^{-1}$ exists, then 0 is not an eigenvalue of $\boldsymbol{A}$. The existence of $\boldsymbol{A}^{-1}$ means that the rank of $\boldsymbol{A}$ is n, which in turn indicates that $\boldsymbol{Ax} = \boldsymbol{0}$ has a unique solution $\boldsymbol{x} = \boldsymbol{0}$. In other words, there is no non-zero $\boldsymbol{x'}$ satisfying $\boldsymbol{Ax'} = \boldsymbol{0x'}$, namely, 0 is not an eigenvalue of $\boldsymbol{A}$. □

Problem 4. Let $\boldsymbol{A}$ be an $n \times n$ square matrix such that $\boldsymbol{A}^{-1}$ exists. Prove: if λ is an eigenvalue of $\boldsymbol{A}$, then $1/\lambda$ is an eigenvalue of $\boldsymbol{A}^{-1}$.

Proof. Since λ is an eigenvalue of $\boldsymbol{A}$, there is a non-zero $n \times 1$ matrix $\boldsymbol{x}$ satisfying

$$\begin{aligned} \boldsymbol{Ax} &= \lambda \boldsymbol{x} \Rightarrow \\ \boldsymbol{A}^{-1}\boldsymbol{Ax} &= \lambda \boldsymbol{A}^{-1}\boldsymbol{x} \Rightarrow \\ \boldsymbol{x} &= \lambda \boldsymbol{A}^{-1}\boldsymbol{x} \Rightarrow \\ \boldsymbol{A}^{-1}\boldsymbol{x} &= (1/\lambda)\boldsymbol{x} \end{aligned}$$

which completes the proof. □

Problem 5. Prove: if $\boldsymbol{A}^2 = \boldsymbol{I}$, then the eigenvalues of $\boldsymbol{A}$ must be 1 or -1.

Proof. Consider any eigenvalue λ of $\boldsymbol{A}$, and let $\boldsymbol{x}$ be an arbitrary eigenvector of $\boldsymbol{A}$ corresponding to λ. Hence, we have:

$$\begin{aligned} \boldsymbol{Ax} &= \lambda \boldsymbol{x} \Rightarrow \\ \boldsymbol{A^2x} &= \lambda \boldsymbol{Ax} \Rightarrow \\ \boldsymbol{Ix} &= \lambda \boldsymbol{Ax} \Rightarrow \\ \boldsymbol{x} &= \lambda \boldsymbol{Ax} \end{aligned}$$

Note that $\lambda(\boldsymbol{Ax}) = \lambda(\lambda \boldsymbol{x}) = \lambda^2 \boldsymbol{x}$. Hence, we have

$$\boldsymbol{x} = \lambda^2 \boldsymbol{x}.$$

As $\boldsymbol{x}$ is not $\boldsymbol{0}$, it follows that $\lambda^2 = 1$, which completes the proof.

Problem 6. Suppose that λ_1 and λ_2 are two distinct eigenvalues of matrix $\boldsymbol{A}$. Furthermore, suppose that $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ under λ_1, and that $\boldsymbol{x}_2$ is an eigenvector of $\boldsymbol{A}$ under λ_2. Prove: there does not exist any real number c such that $c\boldsymbol{x}_1 = \boldsymbol{x}_2$.

Proof. Assume, on the contrary, that such a c exists. Since $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$, we have $\boldsymbol{A}(c\boldsymbol{x}_1) = \lambda_1(c\boldsymbol{x}_1)$, which leads to $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_1 \boldsymbol{x}_2$.

On the other hand, $\boldsymbol{A}\boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$. Therefore, $\lambda_1 = \lambda_2$ (remember $\boldsymbol{x}_2$ cannot be $\mathbf{0}$), giving a contradiction.

Problem 7. Suppose that λ_1 and λ_2 are two distinct eigenvalues of matrix $\boldsymbol{A}$. Furthermore, suppose that $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ under λ_1, and that $\boldsymbol{x}_2$ is an eigenvector of $\boldsymbol{A}$ under λ_2. Prove: $\boldsymbol{x}_1 + \boldsymbol{x}_2$ is *not* an eigenvector of $\boldsymbol{A}$.

Proof. Assume, on the contrary, that $\boldsymbol{x}_1 + \boldsymbol{x}_2$ is an eigenvector under some eigenvalue λ_3. This means that

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}_1 + \boldsymbol{x}_2) &= \lambda_3(\boldsymbol{x}_1 + \boldsymbol{x}_2) \Rightarrow \\
\boldsymbol{A}\boldsymbol{x}_1 + \boldsymbol{A}\boldsymbol{x}_2 &= \lambda_3(\boldsymbol{x}_1 + \boldsymbol{x}_2) \Rightarrow \\
\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2 &= \lambda_3(\boldsymbol{x}_1 + \boldsymbol{x}_2) \Rightarrow \\
(\lambda_1 - \lambda_3)\boldsymbol{x}_{\mathbf{1}} &= (\lambda_3 - \lambda_2)\boldsymbol{x}_{\mathbf{2}}.
\end{aligned}$$

As $\lambda_1 \neq \lambda_2$, at least one of $\lambda_1 - \lambda_3$ and $\lambda_3 - \lambda_2$ is non-zero. Without loss of generality, suppose $\lambda_3 - \lambda_2 \neq 0$, which gives:

$$\frac{\lambda_1 - \lambda_3}{\lambda_3 - \lambda_2}\boldsymbol{x}_{\mathbf{1}} = \boldsymbol{x}_{\mathbf{2}}.$$

In Problem 6, we already showed that the above is impossible, thus giving a contradiction.