Weighted/Prioritised Compensatory Aggregation

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Abstract— **Yager *et al.* first introduce compensatory operators. This paper further introduces a kind of weighted compensatory operators, and a kind of prioritised compensatory operators. The difference between these similar classes of operators are identified. In addition, the paper introduces the concepts of ordered weighted/prioritised compensatory aggregation.**

Keywords—**Compensatory aggregation, weight, priority, fuzzy decision making.**

I. Introduction

Many authors define aggregation as operators generalising "AND" and "OR" fuzzy connectives [3]. However, the two extremal situations of "AND" and "OR" may not always match real-life scenario. Thus, other alternative aggregation operators, such as compensatory operators [4], have been proposed for a tradeoff between these two cases. On the other hand, in traditional aggregation each entity is assumed to carry equal importance. This is not always true. By taking into account the relative importance of each entity, some weighted or prioritised aggregation operators have been proposed, such as weighted mean operators, OWA-operators [5], weighted compensatory aggregations [4] and prioritised disjunction aggregation operators [1], [2]. This paper aims at introducing several new kinds of weighted/prioritised compensatory operators.

The rest of this paper is organised as follows. Sections 2 and 3 introduce the concepts of a kind of weighted compensatory operators and a kind of prioritised compensatory operators, and discuss their basic properties. Section 4 identifies the differences between these two similar classes of operators. Section 5 introduces the concept of ordered weighted/prioritized compensatory operators. Section 6 reviews related work. The final section concludes the paper.

II. Weighted Compensatory Aggregation

First, let us recall the concept of the compensatory operators [4], and the concept of weight operators [4] with respect to compensatory operators.

Definition 1: If an operator $\circ : [0,1] \times [0,1] \to [0,1]$ is increasing, associative and commutative and has unit element $\varepsilon \in (0,1)$, then it is **a compensatory operator**, denoted as $\boxplus^{(\varepsilon)}$. □

Actually, we can regard the unit element of a compensatory operator as a threshold: if an operand is greater than the threshold the operand is regarded as being **positive**; otherwise, the operand is regarded as being **negative**. One of the important properties of compensatory operators is that when one operand is positive and the other is negative, the result of compensatory operation is a tradeoff between the operands [4].

Definition 2: If an operator $\circ : [0,1] \times [0,1] \to [0,1]$ satisfies
1) $\forall a, a' \in [0,1], a \le a' \Rightarrow \omega \circ a \le \omega \circ a'$,
2) $\forall a \in [\varepsilon, 1], \omega \le \omega' \Rightarrow \omega \circ a \le \omega' \circ a$,
3) $\forall a \in [0, \varepsilon], \omega \le \omega' \Rightarrow \omega \circ a \ge \omega' \circ a$,
4) $\forall a \in [0,1], 1 \circ a = a$, and
5) $\forall a \in [0,1], 0 \circ a = \varepsilon$,

where $\varepsilon \in (0,1)$ is a constant, then it is a **weight operator**, denoted as $\diamondsuit^{(\varepsilon)}$. In the **weighted entity** $\omega \diamondsuit^{(\varepsilon)} a$, ω is the **relative weight** (or **weight** for short), and a is the **unweighted entity** (or **entity** for short). □

The axioms on weight operators capture the following intuitions. 1) A weighted entity should increase with the entity. 2) When an entity is positive, the higher its weight, the more positive the weighted entity. 3) When an entity is negative, the higher its weight, the more negative the weighted entity. 4) When an entity has the absolutely highest weight 1, the weighted entity should be the entity itself. 5) An entity with the weight 0 should not effect on the result of the compensatory aggregation. Hence, the weighted entity should be the unit element ε of the compensatory operator $\boxplus^{(\varepsilon)}$.

Second, let us give some properties of weight operators.

Theorem 1:
1) $\varepsilon \le a \le 1 \Rightarrow \varepsilon \le \omega \diamondsuit^{(\varepsilon)} a \le a$,
2) $0 \le a \le \varepsilon \Rightarrow a \le \omega \diamondsuit^{(\varepsilon)} a \le \varepsilon$.

Proof. 1) In the case $\varepsilon \le a \le 1$, by axioms 5, 2 and 4 of weight operators, we have

$$\varepsilon = 0 \diamondsuit^{(\varepsilon)} a \le \omega \diamondsuit^{(\varepsilon)} a \le 1 \diamondsuit^{(\varepsilon)} a = a.$$

2) In the case $0 \le a \le \varepsilon$, by axioms 5, 3 and 4 of weight operators, we have

$$\varepsilon = 0 \diamondsuit^{(\varepsilon)} a \ge \omega \diamondsuit^{(\varepsilon)} a \ge 1 \diamondsuit^{(\varepsilon)} a = a.$$

□

The theorem means that when an entity is positive the weighted entity becomes less positive, and when an entity is negative the weighted entity becomes less negative.

Before defining our weighted compensatory operators, we need one more definition.

Definition 3: If an operator $\circ : [0, +\infty) \times (0, +\infty) \to [0, +\infty)$ satisfies

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1) $a \circ a = 1$,
2) $0 \circ a = 0$,
3) $\omega \circ a$ increases with ω, and
4) $\omega \circ a$ decreases when a increases,
then it is a **general division operator**, denoted as $\oslash$. □

The above concept is a generalisation of the usual arithmetic division operator.

Hereafter unless otherwise specified, the following symbols always take the meaning here: $\vec{a} = (a_1, \cdots, a_n) \in [0,1]^n$, $\vec{w} = (w_1, \cdots, w_n) \in [0,\infty)^n$ such that w_i is the weight of a_i $(0 \le i \le n)$, and $w_{max} = \max\{w_1, \cdots, w_n\}$.

Definition 4: A **weighted compensatory operator (WCO) of dimension** n is a mapping $F_{WCO} : [0,1]^n \times [0,\infty]^n \to [0,1]$, defined as

$$F_{WCO}(\vec{a},\vec{w}) = ((w_1 \oslash w_{max})\Diamond^{(\varepsilon)} a_1) \boxplus^{(\varepsilon)} \cdots \boxplus^{(\varepsilon)}((w_n \oslash w_{max})\Diamond^{(\varepsilon)} a_n). \quad (1)$$

□

Let us discuss the properties of weighted compensatory operators.

Lemma 1: ([4])
1) $\forall a \in [0,\varepsilon), a \boxplus^{(\varepsilon)} 0 = 0$,
2) $\forall a \in (\varepsilon,1], a \boxplus^{(\varepsilon)} 1 = 1$.

Theorem 2:
1) If there is an operand $a_{i_0} = 0$ and its weight $w_{i_0} = w_{max}$, and $\forall i \ne i_0, ((w_i \oslash w_{max})\Diamond a_i) \in [0,\varepsilon)$, then $F_{WCO}(\vec{a},\vec{w}) = 0$.
2) If there is an operand $a_{i_0} = 1$ and its weight $w_{i_0} = w_{max}$, and $\forall i \ne i_0, ((w_i \oslash w_{max})\Diamond a_i) \in (\varepsilon,1]$, then $F_{WCO}(\vec{a},\vec{w}) = 1$.

Proof. When $a_{i_0} = 0$, by axioms 1) and 4) of general division operators we have

$$(w_{i_0} \oslash w_{max})\Diamond a_{i_0} = (w_{max} \oslash w_{max})\Diamond 0 = 1\Diamond 0 = 0.$$

Similarly, when $a_{i_0} = 1$, we have $(w_{i_0} \oslash w_{max})\Diamond a_{i_0} = 1$. Thus, by Lemma 1, when $a_{i_0} = 0$, we have $F_{WCO}(\vec{a},\vec{w}) = 0$; when $a_{i_0} = 1$, we have $F_{WCO}(\vec{a},\vec{w}) = 1$. □

The theorem means: 1) when an entity with the highest weight is absolutely positive and other weighted entities are positive, the result of a WCO aggregation is absolutely positive; and 2) when an entity with the highest weight is absolutely negative and other weighted entities are negative, the result of a WCO aggregation is absolutely negative.

III. Prioritized Compensatory Aggregation

First, we introduce the concept of priority operators with respect to compensatory operators.

Definition 5: If an operator $\circ : [0,1] \times [0,1] \to [0,1]$ satisfies
1) $\forall a, a' \in [0,1], a \le a' \Rightarrow \rho \circ a \le \rho \circ a'$,
2) $\forall a \in [\varepsilon,1], \rho \le \rho' \Rightarrow \rho \circ a \ge \rho' \circ a$,
3) $\forall a \in [0,\varepsilon], \rho \le \rho' \Rightarrow \rho \circ a \le \rho' \circ a$,
4) $\forall a \in [0,1], 1 \circ a = a$,
5) $\forall a \in [\varepsilon,1], 0 \circ a = 1$, and
6) $\forall a \in [0,\varepsilon], 0 \circ a = 0$.

where $\varepsilon \in (0,1)$ is a constant, then it is a **priority operator**, denoted as $\diamond^{(\varepsilon)}$. In the **prioritized entity** $\rho\diamond^{(\varepsilon)}$, ρ is the **relative priority**, and a is the non-prioritized entity (or **entity** for short). □

The five axioms on priority operators capture the following intuitions. 1) A prioritized entity should increase when the entity increases. 2) When an entity is positive, the higher its priority, the less positive the prioritized entity. 3) When an entity is negative, the higher its priority, the less negative the prioritized entity. 4) When an entity has the relative highest priority 1, the prioritized entity is the entity itself. 5) When an entity with the lowest priority 0 is positive, the prioritized entity should become absolutely positive. 6) When an entity with the priority 0 is negative, the prioritized entity should become absolutely negative.

The following theorem reveals some properties of priority operators.

Theorem 3:
1) $\varepsilon \le a \le 1 \Rightarrow 1 \ge \rho \diamond^{(\varepsilon)} a \ge a$,
2) $0 \le a \le \varepsilon \Rightarrow 0 \le \rho \diamond^{(\varepsilon)} a \le a$.

Proof. 1) In the case $\varepsilon \le a \le 1$, by axioms 5, 2 and 4 of priority operators we have

$$1 = 0 \diamond^{(\varepsilon)} a \ge \rho \diamond^{(\varepsilon)} a \ge 1 \diamond^{(\varepsilon)} a = a.$$

2) In the case $0 \le a \le \varepsilon$, by axioms 6, 3 and 4 of priority operators we have

$$0 = 0 \diamond^{(\varepsilon)} a \le \rho \diamond^{(\varepsilon)} a \le 1 \diamond^{(\varepsilon)} a = a.$$

□

The above theorem means that when the entity is positive the prioritized entity is still positive and more positive, whereas when the entity is negative, the prioritized entity is still negative and more negative.

Hereafter unless otherwise specified, the following symbols always take the meaning here: $\vec{\rho} = (\rho_1, \cdots, \rho_n) \in [0,\infty)$ such that ρ_i is the priority of a_i $(0 \le i \le n)$, and $\rho_{max} = \max\{\rho_1, \cdots, \rho_n\}$.

Definition 6: A **prioritized compensatory operator (PCO) of dimension** n is a mapping $F_{PCO} : [0,1]^n \times [0,+\infty)^n \to [0,1]$, defined as

$$F_{PCO}(\vec{a},\vec{\rho}) = ((\rho_1 \oslash \rho_{max}) \diamond^{(\varepsilon)} \rho) \boxplus^{(\varepsilon)} \cdots \boxplus^{(\varepsilon)}((\rho_n \oslash \rho_{max}) \diamond^{(\varepsilon)} a_n). \quad (2)$$

□

Theorem 4:
1) If there is an operand $a_{i_0} = 0$ and its priority $\rho_{i_0} = \rho_{max}$, and $\forall i \ne i_0, ((\rho_i \oslash \rho_{max}) \diamond^{(\varepsilon)} a_i) \in [0,\varepsilon)$, then $F_{PCO}(\vec{a},\vec{\rho}) = 0$.
2) If there is an operand $a_{i_0} = 1$, and its priority $\rho_{i_0} = \rho_{max}$, and $((\rho_i \oslash \rho_{max}) \diamond^{(\varepsilon)} a_i) \in (\varepsilon,1]$, then $F_{PCO}(\vec{a},\vec{\rho}) = 1$.

Proof. The proof is similar to that of Theorem 2. □

The theorem means: 1) when an entity with the highest priority is absolutely positive and the other prioritized entities are positive, the result of a PCO aggregation is absolutely positive; 2) when an entity with the highest priority is absolutely negative and the other prioritized entities are negative, the result of a PCO aggregation is absolutely negative.

IV. Difference between Weight and Priority

The concepts of weight and priority can both be used to indicate the importance level of an entity among a group of entities, but their effects in compensatory aggregation are different. In this section, we will make a theoretic analysis for their difference.

First, we have the following theorem.

Theorem 5:

1) $\varepsilon \leq a \leq 1 \Rightarrow \varepsilon \leq \omega \Diamond^{(\varepsilon)} a \leq a \leq \omega \diamond^{(\varepsilon)} a$,

2) $0 \leq a \leq \varepsilon \Rightarrow 0 \leq \rho \diamond^{(\varepsilon)} a \leq a \leq \rho \Diamond^{(\varepsilon)} a$.

Proof. The result of the theorem is straightforward from Theorems 3 and 1. □

The above theorem reveals the following differences between weight operators and priority operators: 1) When an entity is positive, the weighted entity and the prioritized entity are both positive. The weighted one is less positive than the original one, while the prioritized one is more positive than the original one. 2) When an entity is negative, the weighted entity and the prioritized entity are both negative. The weighted one is less negative than the original one, while the prioritized one is more negative than the original one.

Second, the following table, according to Definitions 2 and 5, further shows us the difference between a weighted operator and a priority operator. That is, given an entity a 1) in the case where a is negative, the weighted entity decreases when the weight ω increases, while the prioritized entity increases when the priority ρ increases; 2) in the case where a is positive, the weighted entity increases when the weight ω increases, while the prioritized entity decreases when the priority ρ increases.

	a	$[0,\varepsilon]$	$[\varepsilon, 1]$
weight operator	$\omega \Diamond^{\varepsilon} a$	↘	↗
priority operator	$\rho \diamond^{\varepsilon} a$	↗	↘

Third, the following theorem reveals the difference between weighted compensatory aggregation and prioritized compensatory aggregation.

Theorem 6: Suppose

$$\vec{a}_* = (a_1, \cdots, a_{i-1}, a, a_{i+1}, \cdots, a_{j-1}, a_j, a_{j+1}, \cdots, a_n),$$
$$\vec{a}^* = (a_1, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_{j-1}, a, a_{j+1}, \cdots, a_n),$$

have the same weight vector and the same priority vector

$$\vec{wp} = (wp_1, \cdots, wp_n).$$

Let $wp_{max} = \max\{wp_1, \cdots, wp_n\}$. And suppose

$$(wp_i \oslash wp_{max}) \circ a_i = (wp_j \oslash wp_{max}) \circ a_j. \quad (3)$$

1) When $a \in [\varepsilon, 1]$, if $wp_i \leq wp_j$, then

$$F_{WCO}(\vec{a}_*, \vec{wp}) \leq F_{WCO}(\vec{a}^*, \vec{wp}), \quad (4)$$
$$F_{PCO}(\vec{a}_*, \vec{wp}) \geq F_{PCO}(\vec{a}^*, \vec{wp}). \quad (5)$$

2) When $a \in [0, \varepsilon]$, if $wp_i \leq wp_j$, then

$$F_{WCO}(\vec{a}_*, \vec{wp}) \geq F_{WCO}(\vec{a}^*, \vec{wp}), \quad (6)$$
$$F_{PCO}(\vec{a}_*, \vec{wp}) \leq F_{PCO}(\vec{a}^*, \vec{wp}). \quad (7)$$

Proof. We only prove (4). Others can be proved similarly. From the assumption of the theorem, for any $k \neq i, j$, we have

$$(wp_k \oslash wp_{max}) \Diamond^{(\varepsilon)} a_k = (wp_k \oslash wp_{max}) \Diamond^{(\varepsilon)} a_k.$$

Again noticing (3), we have

$$\begin{aligned} & \boxplus^{(\varepsilon)}\{(wp_k \oslash wp_{max}) \Diamond^{(\varepsilon)} ap_k \mid k \neq i\} \\ = \; & \boxplus^{(\varepsilon)}\{(wp_k \oslash wp_{max}) \Diamond^{(\varepsilon)} a_k \mid k \neq j\}. \end{aligned}$$

In the case $a \in [\varepsilon, 1]$, by property 2 of weight operators and from condition $wp_i \leq wp_j$, we can obtain

$$(wp_i \oslash wp_{max}) \Diamond^{(\varepsilon)} a \leq (wp_j \oslash wp_{max}) \Diamond^{(\varepsilon)} a.$$

Thus, by the commutativity and monotonicity of compensatory operators, (4) holds. □

The above theorem reveals the different effects of weights and priorities in compensatory aggregation: 1) when an entity is positive, the higher its importance, the bigger the result of weighted compensatory aggregation tends to be, whereas the smaller the result of prioritized compensatory aggregation tends to be; and 2) when an entity is negative, the higher its importance, the smaller the result of weighted compensatory aggregation tends to be, whereas the bigger the result of prioritized compensatory aggregation tends to be.

V. Ordered Weighted/Prioritized Compensatory Aggregation

Definition 7: An **ordered weighted compensatory operator (OWCO) of dimension** n is a mapping $F_{OWCO} : [0,1]^n \times [0,1]^n \to [0,1]$, defined as

$$F_{OWCO}(\vec{a}, \vec{w}) = ((w_1 \oslash w_{max}) \Diamond^{(\varepsilon)} a_{\sigma(1)}) \boxplus^{(\varepsilon)} \cdots \boxplus^{(\varepsilon)} ((w_n \oslash w_{max}) \Diamond^{(\varepsilon)} a_{\sigma(n)}); \quad (8)$$

an **ordered prioritized compensatory operator (OPCO) of dimension** n is a mapping $F_{OPCO} : [0,1]^n \times [0,1]^n \to [0,1]$, defined as

$$F_{OPCO}(\vec{a}, \vec{\rho}) = ((\rho_1 \oslash \rho_{max}) \diamond^{(\varepsilon)} a_{\sigma(1)}) \boxplus^{(\varepsilon)} \cdots \boxplus^{(\varepsilon)} ((\rho_n \oslash \rho_{max}) \diamond^{(\varepsilon)} a_{\sigma(n)}), \quad (9)$$

where $\{a_{\sigma(1)}, \cdots, a_{\sigma(n)}\}$ is a permutation of $\{a_1, \cdots, a_n\}$ such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$ $(1 \leq i \leq n-1)$, $\vec{w} = (w_1, \cdots, w_n)$ such that w_i is the weight of ordered position i, and $\vec{\rho} = (\rho_1, \cdots, \rho_n)$ such that ρ_i is the priority of ordered position i $(1 \leq i \leq n)$. □

VI. Related Work

The definition of weighted compensatory operators improves upon that first introduced by Yager *et al.* [4]. In our operators weights take values on $[0, \infty)$, while in Yager's operators weights take values on $[0, 1]$. In other words, when using Yager's operators users have to assign maximum value 1 to every entity with the highest importance. In a dynamic environment, entities may be added or deleted during aggregation. What if an entity with an importance higher than that of any existing entity is added in? And what if the entity with maximum weight 1 is removed? This might involve recalculating the weights of all entities. However, no such service is provided in Yager's operators. Thus, users have to re-assign weights to all entities again. This might cause trouble to users, especially in the case of a large number of entities. In our framework there is no maximum value for weights, and so a bigger value can be assigned to an added entity if it is considered to be more important than any existing entity. Similarly, we can understand the situation in which some entities are deleted.

Dubois and Prade [1], and Luo, Leung and Lee [2] propose prioritized disjunction aggregation operators. Disjunction aggregation operators (T-norms) and compensatory operators are special cases of uninorms [4]. The idea behind prioritized disjunction aggregation operators is employed to introduce prioritized compensatory operators in this paper. In addition, the idea behind ordered weighted average operator of Yager [5] is used to define ordered weighted/prioritized compensatory operators.

VII. Conclusion

This paper introduces concepts of several kinds of weighted/prioritized compensatory operators, discusses their priorities, and identifies the differences between weighted compensatory operators and prioritized compensatory operators.

Acknowledgement

The work described in this paper was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (RGC Ref. No. CUHK4304/98E).

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