A Spectrum of Compensation Aggregation Operators

Xudong Luo[‡], Jimmy Ho-man Lee and Ho-fung Leung

Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, N.T., Hong Kong SAR, P.R. China
Email: {xluo,jlee,lhf}@cse.cuhk.edu.hk

Abstract—**In a decision process, when aggregating two values with conflict meaning, sometimes the result should be a tradeoff between the two values. Applicable to many real problems, compensation operators are aggregation operators with such a property. In order to offer more freedom in the selection of suitable compensation operators for various specific application, this paper explores new sorts of compensation operators. First, we introduce the concept of general compensation operators, which form a subclass of general aggregation operators. The two existing kinds of compensation operators, compensatory operators (a special case of uninorm operators) and averaging operators, are subclasses of our general compensation operators. Second, we identify seven new subclasses of the general compensation operators. Third, we construct a new kind of compensation operator, the gray averaging operator, which can include T-norms, T-conorm and averaging operators as its special cases.**

Keywords—**Fuzzy decision making, aggregation operator, compensation operator.**

I. Introduction

Many researchers define aggregation as operators generalising "AND" and "OR" fuzzy connectives [8]. However, the two extremal situations of "AND" and "OR" may not always be able to match real-life scenario. Thus, compensation operators, which can obtain a tradeoff between these two extremal situation, have been proposed, such as averaging operators [1], OWA-operators [9] and compensatory operators [12], [3].

Compensation operators are applicable in many real problems, *e.g.*, automated negotiation problems [5] in e-commerce, meeting scheduling problems [6], solution synthesis [13] in distributed expert systems, and parallel combination [7] of uncertainties in expert systems. The best choice of the operators may vary from problem to problem. In order to offer more freedom in selecting suitable compensation operators for various specific applications, the paper aims at unveiling new compensation operators and their respective properties.

The rest of this paper is organised as follows. Section 2 recalls the concepts of two kinds of compensation operators, and general aggregation operators. Section 3 defines general compensation operators which form a subclass of general aggregation operators. The two existing kinds of compensation operators are subclasses of our general compensation operators. Our framework captures also the seven new subclasses identified in Section 4. Section 5 constructs a new kind of compensation operator: the gray averaging operator. The last section concludes the paper.

[‡]The current correspondence address: Department of Electronics and Computer Science, The University of Southampton, Highfield, Southampton, SO17 1BJ, United Kingdom, Email: xl@ecs.soton.ac.uk.

II. Preliminaries

This section recalls concepts of two kinds of compensation operators and general aggregation operators.

The first kind of compensation operators are compensatory operators—a special case of uninorm operators [12], [2], [10], [11], [3], which is defined as follows [12]:

Definition 1: A binary operator $\circ : [0,1] \times [0,1] \rightarrow [0,1]$ is a **uninorm** if it is increasing, associative and commutative and has unit element $\varepsilon \in [0,1]$. In particular, when ε is 1, 0, and between 1 and 0, respectively, a uninorm operator with unit element ε is called a **T-norm** (denoted as $\triangle$), a **T-conorm** (denoted as $\bigtriangledown$), and a **compensatory operator** (denoted as $\boxplus^{(\varepsilon)}$) respectively. □

The following lemma gives an important property of compensatory operators.

Lemma 1: A compensatory operator $\circ$ with unit element ε has the following properties:

$$\varepsilon_1^+ \circ \varepsilon_2^+ \geq \max\{\varepsilon_1^+, \varepsilon_2^+\}, \tag{1}$$

$$\varepsilon_1^- \circ \varepsilon_2^- \leq \min\{\varepsilon_1^-, \varepsilon_2^-\}, \tag{2}$$

$$\varepsilon_1^- \leq \varepsilon_1^- \circ \varepsilon_2^+ \leq \varepsilon_2^+, \tag{3}$$

where $\varepsilon_1^-, \varepsilon_2^- \in [0, \varepsilon)$, $\varepsilon_1^+, \varepsilon_2^+ \in (\varepsilon, 1]$, and $\varepsilon \in (0,1)$ is a constant (hereafter if no confusion occurs the symbols ε, ε_1^-, ε_2^-, ε_1^+ and ε_2^+ always take the meaning here). □

Actually, we can regard the unit element of a compensatory operator as a threshold: if an evaluation is greater than the threshold the evaluation is regarded as being **positive**; otherwise, the evaluation is regarded as being **negative**. Thus, in Lemma 1, property (1) reveals the intuition that when two evaluations are both positive they should enhance the effect of each other; property (2) reveals the intuition that when two evaluations are both negative, they should weaken each other; and property (3) means that when two evaluations are in conflict, we should get a compromise.

Two important properties of T-norms and T-conorms, which will be used later, are as follows.

Lemma 2:

$$\triangle(a_1, \cdots, a_n) \leq \min\{a_1, \cdots, a_n\}$$

$$\leq \max\{a_1,\cdots,a_n\} \leq \nabla(a_1,\cdots,a_n), \quad (4)$$

$$\Delta(a,\cdots,a) = \nabla(a,\cdots,a) = a. \quad (5)$$

□

The second kind of compensation operators are the averaging operators defined as follows [1]:

Definition 2: A function $M : [0,1]^n \leftarrow [0,1]$ is an **averaging operator** if

1) idempotency: $\forall a \in [0,1], M(a,\cdots,a) = a$;

3) monotonicity: if $a_1 \leq a'_1,\cdots,a_n \leq a'_n$, $M(a_1,\cdots,a_n) \leq M(a'_1,\cdots,a'_n)$,

2) symmetry: $M(a_1,\cdots,a_n) = M(a'_1,\cdots,a'_n)$, where $(a'_1,\cdots,a'_n)$ is any permutation on $\{a_1,\cdots,a_n\}$;

□

The following lemma gives an important property of averaging operators.

Lemma 3: $\forall a_1,\cdots,a_n \in [0,1]$,

$$\min\{a_1,\cdots,a_n\} \leq M(a_1,\cdots,a_n) \leq \max\{a_1,\cdots,a_n\}. \quad (6)$$

□

This lemma means that an averaging operator M must give a result lying between the minimum and maximum among M's operands.

In fuzzy mathematics, a general aggregation operator is defined as follows [4].

Definition 3: A function $h : [0,1]^n \to [0,1]$ is a **general aggregation operator** if

1) boundary conditions: $h(0,\cdots,0) = 0$ and $h(1,\cdots,1) = 1$;

2) monotonicity: if $a_1 \leq a'_1,\cdots,a_n \leq a'_n$, then $h(a_1,\cdots,a_n) \leq h(a'_1,\cdots,a'_n)$;

3) symmetry: $h(a_1,\cdots,a_n) = h(a'_1,\cdots,a'_n)$, where $(a'_1,\cdots,a'_n)$ is any permutation on $\{a_1,\cdots,a_n\}$.

□

Clearly, uninorm operators and averaging operators are special cases of the general aggregation operators. Of course, as special cases of uninorms, T-norms, T-conorms and compensatory operators are also special cases of the general aggregation operators.

III. Basic Concept

In a decision process, the idea behind tradeoffs is to regard the evaluation of an action as lying between the *worst* and the *best* local ratings. This occurs in the presence of conflicting goals, when a compensation between the corresponding compatibilities is allowed. Both compensatory operators and averaging operators can realize tradeoffs between objectives, by allowing a compensation between ratings. Through analysing the concepts of compensatory operators, averaging operators and general aggregation operators, we define a kind of general compensation operators as follows.

Definition 4: A binary operator $\circ : [0,1] \times [0,1] \to [0,1]$ is a **general compensation operator** if it is increasing, associative and commutative, and satisfies

$$\min\{\varepsilon_1^-,\varepsilon_2^+\} \leq \varepsilon_1^- \circ \varepsilon_2^+ \leq \max\{\varepsilon_1^-,\varepsilon_2^+\}. \quad (7)$$

□

The following theorem reveals the relationship between general compensation operators and general aggregation operators, compensatory operators as well as averaging operators.

Theorem 1: A general compensation operator is a general aggregation operator. Both compensatory operators and averaging operators are general compensation operators.

Then, do our general compensation operators have other subclasses? The answer is affirmative. Clearly, the following four points

$$l^- = \min\{\varepsilon_1^-,\varepsilon_2^-\}, \quad u^- = \max\{\varepsilon_1^-,\varepsilon_2^-\},$$
$$l^+ = \min\{\varepsilon_1^+,\varepsilon_2^+\}, \quad u^+ = \max\{\varepsilon_1^+,\varepsilon_2^+\},$$

and point ε divide the interval $[0,1]$ into the following six subintervals:

$$[0,l^-], \ [l^-,u^-], \ [u^-,\varepsilon], \ [\varepsilon,l^+], \ [l^+,u^+], \ [u^+,1].$$

(1) and (2) just mean that $\varepsilon_1^+ \circ \varepsilon_2^+$ and $\varepsilon_1^- \circ \varepsilon_2^-$ fall into $[0,l^+]$ and $[u^-,1]$ respectively. All combinations of intervals, which $\varepsilon_1^- \circ \varepsilon_2^-$ and $\varepsilon_1^+ \circ \varepsilon_2^+$ could fall into, are exhaustively enumerated in the following table:

$([0,l^-],[\varepsilon,l^+])$	$([0,l^-],[l^+,u^+])$	$([0,l^-],[u^+,1])$
$([l^-,u^-],[\varepsilon,l^+])$	$([l^-,u^-],[l^+,u^+])$	$([l^-,u^-],[u^+,1])$
$([u^-,\varepsilon],[\varepsilon,l^+])$	$([u^-,\varepsilon],[l^+,u^+])$	$([u^-,\varepsilon],[u^+,1])$

Clearly, by Lemma 1 compensatory operators correspond to the case of $([0,l^-],[u^+,1])$; by Lemma 3 averaging operators correspond to the case of $([l^-,u^-],[l^+,u^+])$. In the following section, we will introduce seven new subclass of general compensation operators, which correspond to the other seven cases respectively.

IV. Concepts of Seven New Classes of Compensation Operators

In this section, we identify seven subclasses of general compensation operators.

Definition 5: A general compensation operator is an **optimistic compensation operator**, denoted as $\boxdot^{(\varepsilon)}$, if

$$\varepsilon_1^+ \boxdot^{(\varepsilon)} \varepsilon_2^+ \geq \max\{\varepsilon_1^+,\varepsilon_2^+\}, \quad (8)$$
$$\varepsilon_1^- \boxdot^{(\varepsilon)} \varepsilon_2^- \geq \max\{\varepsilon_1^-,\varepsilon_2^-\}. \quad (9)$$

□

In the above definition, (8) captures the intuition that when two evaluations are both positive they should enhance the effect of each other; (9) captures the intuition that when two evaluations are both negative, they also enhance each other. In real life, the combined situation of (8) and (9) could happen. For example, there could be persons who are always optimistic no matter if they are in a good or a bad situation. The combination of (8) and (9) can capture the attitude of such persons.

Definition 6: A general compensation operator is a **pessimistic compensation operator**, denoted as $\boxminus^{(\varepsilon)}$, if

$$\varepsilon_1^+ \boxminus^{(\varepsilon)} \varepsilon_2^+ \leq \min\{\varepsilon_1^+,\varepsilon_2^+\}, \quad (10)$$
$$\varepsilon_1^- \boxminus^{(\varepsilon)} \varepsilon_2^- \leq \min\{\varepsilon_1^-,\varepsilon_2^-\}. \quad (11)$$

□

Pessimistic compensation operators are exact converse of optimistic compensation operators. (10) and (11) capture the intuition that the evaluation weaken the effect of each other no matter if both of them are positive or negative. In real life, the combined situation of (10) and (11) could happen. For example, there could be persons who are always pessimistic no matter if they are in a good or a bad situation. The combination of (10) and (11) can capture the attitude of such persons.

Definition 7: A general compensation operator is an **odd compensation operator**, denoted as $\boxtimes^{(\epsilon)}$, if

$$\varepsilon_1^+ \boxtimes^{(\epsilon)} \varepsilon_2^+ \leq \min\{\varepsilon_1^+, \varepsilon_2^+\}, \quad (12)$$
$$\varepsilon_1^- \boxtimes^{(\epsilon)} \varepsilon_2^- \geq \max\{\varepsilon_1^-, \varepsilon_2^-\}. \quad (13)$$

□

In the above definition, (12) captures the intuition that when two evaluations are both positive they should weaken the effect of each other; (13) captures the intuition that when two evaluations are both negative, they should enhance each other. In real life, the combined situation of (12) and (13) could happen. For example, there are some kind of persons: when they are in a good situation they do things cautiously, while when they are in a bad situation they do not lose their confidence and so do things boldly.

Similarly, we can understand the following definition.

Definition 8: Let operator $\circ$ is a general compensation operator. Operator $\circ$ is an **odd positive compensation operator** if

$$\min\{\varepsilon_1^+, \varepsilon_2^+\} \leq \varepsilon_1^+ \circ \varepsilon_2^+ \leq \max\{\varepsilon_1^+, \varepsilon_2^+\}, \quad (14)$$
$$\varepsilon_1^- \circ \varepsilon_2^- \geq \max\{\varepsilon_1^-, \varepsilon_2^-\}. \quad (15)$$

Operator $\circ$ is a **pessimistic positive compensation operator** if

$$\min\{\varepsilon_1^+, \varepsilon_2^+\} \leq \varepsilon_1^+ \circ \varepsilon_2^+ \leq \max\{\varepsilon_1^+, \varepsilon_2^+\}, \quad (16)$$
$$\varepsilon_1^- \circ \varepsilon_2^- \leq \min\{\varepsilon_1^-, \varepsilon_2^-\}. \quad (17)$$

Operator $\circ$ is an **optimistic negative compensation operator** if

$$\varepsilon_1^+ \circ \varepsilon_2^+ \geq \max\{\varepsilon_1^+, \varepsilon_2^+\}, \quad (18)$$
$$\min\{\varepsilon_1^-, \varepsilon_2^-\} \leq \varepsilon_1^- \circ \varepsilon_2^- \leq \max\{\varepsilon_1^-, \varepsilon_2^-\}. \quad (19)$$

Operator $\circ$ is a **pessimistic positive compensation operator** if

$$\varepsilon_1^+ \circ \varepsilon_2^+ \leq \min\{\varepsilon_1^+, \varepsilon_2^+\}, \quad (20)$$
$$\min\{\varepsilon_1^-, \varepsilon_2^-\} \leq \varepsilon_1^- \circ \varepsilon_2^- \leq \max\{\varepsilon_1^-, \varepsilon_2^-\}. \quad (21)$$

□

V. Gray Averaging Operator

In previous section, we just introduce some concepts of various compensation operators. This section will construct a specific family of compensation operators.

Definition 9: A function $GM : [0,1]^n \to [0,1]$, defined as follows, is called a **gray averaging operator** with parameters $\diamondsuit \in \{\wedge, \vee\}$ and $\zeta \in (0,1)$:

$$GM(\vec{a}, \diamondsuit, \zeta) = \begin{cases} X + \zeta \times (M(X,Y) - X) & \text{if } \diamondsuit = \wedge, \\ Y - \zeta \times (Y - M(X,Y)) & \text{if } \diamondsuit = \vee, \end{cases} \quad (22)$$

where $\vec{a} \in [0,1]^n$, $X = \triangle(\vec{a})$, and $Y = \nabla(\vec{a})$. ζ is called a **gray degree** of GM. □

Theorem 2: Gray averaging operators are averaging operators.

Proof. 1) By Lemma 2, property 1 of Definition 2 holds for a gray averaging operator. 2) By Definition 1, $\triangle$ and ∇ satisfy commutativity and associativity, and so they are symmetric. Thus, a gray averaging operator is also symmetric. 3) By Definitions 1 and 2, $\triangle$, ∇ and GM are increasing, and thus by (22) a gray averaging operator is also increasing. □

The following theorem reveals the essential idea behind gray averaging operators.

Theorem 3: Let $X = \triangle(\vec{a})$, and $Y = \nabla(\vec{a})$, where each $\vec{a} \in [0,1]^n$. Then

$$\begin{aligned} X &\leq X + \zeta \times (M(X,Y) - X) \\ &\leq M(X,Y) \\ &\leq Y - \zeta \times (Y - M(X,Y)) \leq Y, \end{aligned} \quad (23)$$

where $\zeta \in [0,1]$, and M is an averaging operator.

Proof. By Lemmas 2 and 3, we have

$$X \leq M(X,Y) \leq Y.$$

Since $\zeta \in [0,1]$, (23) holds. □

The above theorem reveals: 1) When $\diamondsuit = \wedge$, $1 - \zeta$ can be viewed as a degree to which the relationship between operands of the gray averaging operator with parameters $\diamondsuit$ and ζ is a disjunction relationship. In particular, when $\diamondsuit = \wedge$ and $1 - \zeta = 1$, the relationship is exactly a disjunction relationship since the gray averaging operator degenerates to a T-norm which always corresponds to disjunction. 2) When $\diamondsuit = \vee$, $1 - \zeta$ can be viewed as a degree to which the relationship between operands of the gray averaging operator with parameters $\diamondsuit$ and ζ is exactly a conjunction relationship. In particular, when $\diamondsuit = \vee$ and $1 - \zeta = 1$, the relationship is a conjunction since the gray averaging operator degenerates to a T-conorm which always corresponds to conjunction. 3) As long as $\zeta = 1$, the gray averaging operator is equal to an averaging operator M.

One class of averaging operations that covers the entire interval between the min and max operations consists of generalised means [4] defined as

$$h_\alpha(a_1, \cdots, a_n) = \left(\frac{a_1^\alpha + \cdots + a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}, \quad (24)$$

where $\alpha \in (-\infty, 0) \cup (0, \infty)$ is a parameter by which different means are distinguished. Unlike our gray model, this function is not able to cover the entire interval between any pair of T-norms and T-conorms.

VI. Conclusion

By analysing two kinds of compensation operators and general aggregation operators, we define general compensation operators. The concept can cover not only two existing kinds of compensation operators but also seven new kinds of compensation operators. We also construct the gray averaging operator. The operator can cover the entire interval between any pair of T-norm and T-conorm, while a previous averaging operator just can do so between a special pair of T-norm and T-conorm operations, min and max. Besides, the concept of the gray averaging operators can include T-norms, T-conorms and averaging operators as its special cases. Uninorms are less general than the gray averaging operators since it fails to include averaging operators.

The following is worth further pursuit: 1) to construct more specific compensation operators, 2) to apply various specific compensation operators to real problems, and evaluate their usefulness in practical applications.

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