Introduction to Game Theory and its Applications in Computer Networks: Introduction Part II

John C.S. Lui[1] Daniel R. Figueiredo[2]

[1]Department of Computer Science & Engineering
The Chinese University of Hong Kong

[2]School of Computer & Communication Sciences
Swiss Federal Institute of Technology-Lausanne (EPFL)

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Outline

1. **Dynamic Game of Complete Information**
    - Dynamic Games of Complete & Perfect Information
    - Two-Stage Dynamic Games of Complete but Imperfect Information
    - Repeated Games
2. **Static Games of Incomplete Information**
    - Theory: Static Bayesian Games and Bayesian NE

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   - Two-Stage Dynamic Games of Complete but Imperfect Information
   - Repeated Games
2. Static Games of Incomplete Information
   - Theory: Static Bayesian Games and Bayesian NE

Basic Theory: Backwards Induction

Definition

- Dynamic Game: Game in which we have sequence of moves.
- Complete Information: Games in which the strategy space and player's payoff functions are common knowledge.
- Perfect Information: Each move in the game the player with the move knows the full *history* of the game thus far.
- Imperfect Information: At some move the player with the move does not know the history of the game.

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- Consider a game in which
  1. Player 1 chooses an action a_1 from the feasible set A_1.
  2. Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2.
  3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.
- **Example:** Player 1 chooses between giving player 2 $1,000 or nothing. Player 2 observes player 1's move, and choose to explode a grenade that will kill both players.
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Solution Technique: Backwards Induction

- When player 2 gets the move, he needs to solve

$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$

  Assume that for each action a_1, the above optimization problem has a unique solution, denoted by $R_2(a_1)$ (which is player 2's *best response*).

- Then, player 1 needs to solve:

$$\max_{a_1 \in A_1} u_2(a_1, R(a_1)).$$

- Assume this optimization has a unique solution a_1^*, the backwards-induction outcome of this game is $(a_1^*, R_2(a_1^*))$.
- Compare with the previous *normal-form* representation of a game, the verbal description of (1) – (3) is called the extensive-form representation of the game.

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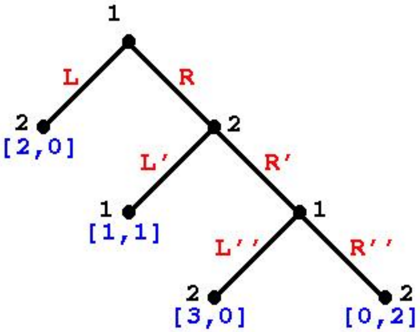
Extensive Form Representation



Playing this game

- In round 3, player 1 chooses L''.
- In round 2, player 2 chooses L'.
- In round 1, player 1 chooses L. Thus the game ends in the first round.

Stackelberg Game

Definition (Stakelberg Game)

- Two players in this game: a leader and a follower.
- The leader moves first, choosing a strategy.
- Then the follower observes the leader's choice and picks a strategy.

- Under the Stackelberg game, the leader chooses strategy knowing that the follower will apply best response.
- Every Stackelberg equilibrium is also Subgame Perfect Nash Equilibrium.

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Stackelberg Model of Duopoly

Example

Consider two firms where

- firm 1 chooses quantity $q_1 \geq 0$,
- firm 2 observes q_1, chooses quantity $q_2 \geq 0$,
- firm i profit function

$$\pi_i(q_i, q_j) = q_i \left[P(Q) - c\right].$$

where $P(Q) = a - Q$ is the market clearing price and $Q = q_1 + q_2$, the aggregate quantity and c is a constant.

Stackelberg Duopoly and Backward Induction

- Firm 2's solution is $R_2(q_1)$, :

$$\begin{aligned} \max_{q_2 \geq 0} \pi_2(q_1, q_2) &= \max_{q_2 \geq 0} q_2 \left[a - q_1 - q_2 - c \right], \\ R_2(q_1) &= \frac{a - q_1 - c}{2}, \quad \text{where } q_1 < a - c. \end{aligned}$$

- Firm 1's response:

$$\max_{q_1 \geq 0} \pi_1(q_1, R_2(q_1)) = \max_{q_1 \geq 0} q_1 \frac{a - q_1 - c}{2},$$

$$q_1^* = \frac{a-c}{2}; \qquad R_2(q_1^*) = \frac{a-c}{4}.$$

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$$q_1^* = \frac{a - c}{2}; \qquad R_2(q_1^*) = \frac{a - c}{4}.$$

Comment on Stackelberg Duopoly

- Recall that the Nash equilibrium in the Cournot game, each firm produces $(a-c)/3$.
- The aggregate quantity of the Stackelberg game, $3(a-c)/4$, is greater than the Nash equilibrium of the Cournot game $2(a-c)/3$.
- The market-clearing price is lower in the Stackelberg game.
- Firm 1's profit in the Stackelberg game is higher than its profit in the Cournot game.
- Firm 1's better off in the Stackelberg game implies firm 2 is worse off. (Leader's advantage).

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Sequential Bargaining: 3-period, 1 unit of resource

- In the first period, Player 1 proposes to take s_1 of the resource, leaving $1 - s_1$ to Player 2.
- Player 2 either **accepts** (and the game ends with payoffs s_1 to Player 1 and $1 - s_1$ to Player 2), or **reject** (the game continues).
- In the second period, Player 2 proposes that Player 1 to take s_2 of the resource, leaving $1 - s_2$ to Player 2.
- Player 1 either **accepts** (and the game ends with payoffs s_2 to Player 1 and $1 - s_2$ to Player 2), or **reject** (the game continues).
- In the third period, Player 1 receives s of the resource, player 2 receives $1 - s$ of the resource, where $0 < s < 1$.

There is a **discount** factor δ per period, $0 < \delta < 1$.

Solution

- Consider Player 2's optimal offer if the 2nd period is reached.
- Player 1 is facing a choice, choose s_2 or receive δs. Player 1 will accept the offer **iff**

$$s_2 \geq \delta s.$$

- Player 2's 2nd-period decision:
  1. receiving $1 - \delta s$ (by offering $s_2 = \delta s$ to Player 1), or
  2. receiving $\delta(1 - s)$ in the third period.
- Since $1 - \delta s > \delta(1 - s)$, Player 2's optimal 2nd-round choice is $s_2^* = \delta s$ and Player 1 will accept.

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Solution: continue

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  1. $1 - s_1 \geq \delta(1 - s_2^*)$, or
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The solution of the game should end in the 1st-period with $(s_1^*, 1 - s_1^*)$, where $s_1^* = 1 - \delta(1 - \delta s)$.

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Extension to infinite rounds

- What about if we have *infinite* number of rounds?
- Truncate the infinite-horizon game and apply the logic from the finite-horizon case.
- The game in the 3rd period, should it be reached, is identical to the game beginning in the 1st period.
- Let S_H be the highest payoff player 1 can achieve in any backwards-induction outcome of the game as a whole.

Extension to infinite rounds: continue

- Using S_H as the 3rd period payoff to player 1.
- Player 1's first-period payoff is $f(S_H)$ where

$$f(s) = 1 - \delta + \delta^2 s.$$

- But S_H is also the highest possible 1st-period payoff, so $f(S_H) = S_H$.
- The only value of s that satisfy $f(s) = s$ is

$$s^* = 1/(1 + \delta).$$

- Solution is, in the first round, player 1 offers $(s^*, 1 - s^*) = (1/(1 + \delta), \delta/(1 + \delta))$ to player 2, who will accept.

Framework

- We allow simultaneous moves (which corresponds to "*imperfect information*") with each stage.
- Consider the following two-stage game:
  1. Player 1 and 2 simultaneously choose action $a_1 \in A_1$ and $a_2 \in A_2$ respectively.
  2. Player 3 and 4 observe the outcome of the 1st stage (a_1, a_2), then simultaneously choose action $a_3 \in A_3$ and $a_4 \in A_4$ respectively.
  3. Payoffs are $u_i(a_1, a_2, a_3, a_4)$ for $i = 1, 2, 3, 4$.
- Various of the above game (1) players 3 and 4 are player 1 and 2; (2) player 2 or player 4 is missing.

Framework: continue

- For each outcome of (a_1, a_2), the 2nd stage game has a unique Nash equilibrium, denoted by $(a_3^*(a_1, a_2), a_4^*(a_1, a_2))$ (assumption of NE).
- Player 1 and 2 anticipate $(a_3^*(a_1, a_2), a_4^*(a_1, a_2))$, then both players simultaneously take action with the payoff of $u_i(a_1, a_2, a_3^*(a_1, a_2), a_4^*(a_1, a_2))$ for $i = 1, 2$.
- Suppose (a_1^*, a_2^*) is the unique Nash equilibrium, then

$$(a_1^*, a_2^*, a_3^*(a_1^*, a_2^*), a_4^*(a_1^*, a_2^*))$$

is the **subgame-perfect outcome**.

Bank Runs

- Two investors deposited D with a bank.
- The bank invested in a project. If it liquidate before the project matures, a total return of $2r$, where $D > r > D/2$. If the project matures, a total return of $2R$, where $R > D$.
- Investors can withdraw on date 1 (before the project matures) or date 2 (after the project matures).
- The game is:
  1. If both investors make withdrawals at date 1, each receives r, game ends.
  2. If only one makes withdrawal at date 1, that investor receives D, other receives $2r - D$, game ends.
  3. If both withdraw at date 2, each receives R, game ends.
  4. If only one withdraws at date 2, that investor receives $2R - D$, other receives D, game ends.
  5. If neither makes withdrawal at date 2, banks returns R to each investor, game ends.

"Normal-Form" of the game

For two dates

Date 1	Investor 2 **(Withdraw)**	Investor 2 **(Don't)**
Investor 1 **(Withdraw)**	r,r	D, 2r-D
Investor 1 **(Don't)**	2r-D, D	next stage

Date 2	Investor 2 **(Withdraw)**	Investor 2 **(Don't)**
Investor 1 **(Withdraw)**	R,R	2R-D,D
Investor 1 **(Don't)**	D, 2R-D	R,R

Analysis

- Consider date 2, since $R > D$ (and so $2R - D > D$ and $2R - D > R$), "**withdraw**" strictly dominates "**Don't**", we have a unique Nash equilibrium.
- For date 1, we have:

Date 1	Investor 2 (Withdraw)	Investor 2 (Don't)
Investor 1 (Withdraw)	r,r	D, 2r-D
Investor 1 (Don't)	2r-D, D	R,R

- Since $r < D$ (and so $2r - D < r$), we have two pure-strategy Nash Equilibrium, (a) both withdraw, (b) both don't withdraw, with the 2nd NE being efficient.

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Tariffs and Imperfect Competition

- Consider two countries, denoted by $i = 1, 2$, each setting a tariff rate t_i per unit of product.
- A firm produces output, both for home consumption and export.
- Consumer can buy from a local firm or foreign firm.
- The market clearing price for country i is $P(Q_i) = a - Q_i$, where Q_i is the quantity on the market in country i.
- A firm in i produces $h_i(e_i)$ units for local (foreign) market, i.e., $Q_i = h_i + e_j$.
- The production cost of firm i is $C_i(h_i, e_i) = c(h_i + e_i)$ and it pays $t_j e_i$ to country j.

Tariffs and Imperfect Competition Game

- First, the government *simultaneously* choose tariff rates t_1 and t_2.
- Second, the firms observe the tariff rates, decide (h_1, e_1) and (h_2, e_2) *simultaneously*.
- Third, payoffs for both firms and governments:
  (1) Profit for firm i:

$$\pi_i(t_i, t_j, h_i, e_i, h_j, e_j) = [a - (h_i + e_j)]h_i + [a - (e_i + h_j)]e_i - c(h_i + e_i) - t_j e_i$$

  (2) Welfare for government i:

$$W_i(t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j$$

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$$\begin{aligned}\pi_i(t_i, t_j, h_i, e_i, h_j, e_j) &= [a - (h_i + e_j)]h_i + [a - (e_i + h_j)]e_i \\ &\quad - c(h_i + e_i) - t_j e_i\end{aligned}$$

  (2) Welfare for government i:

$$W_i(t_i, t_j, h_i, e_i, h_j, e_j) = \frac{1}{2}Q_i^2 + \pi_i(t_i, t_j, h_i, e_i, h_j, e_j) + t_i e_j$$

Tariffs and Imperfect Competition Game

- First, the government *simultaneously* choose tariff rates t_1 and t_2.
- Second, the firms observe the tariff rates, decide (h_1, e_1) and (h_2, e_2) *simultaneously*.
- Third, payoffs for both firms and governments:
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Tariffs and Imperfect Competition Game: 2nd stage

- Suppose the governments have chosen t_1 and t_2.
- If $(h_1^*, e_1^*, h_2^*, e_2^*)$ is a NE for firm 1 and 2, firm i needs to solve $\max_{h_i, e_i \geq 0} \pi_i(t_i, t_j, h_i, e_i, h_j^*, e_j^*)$. After re-arrangement, it becomes two **separable** optimizations:

$$\max_{h_i \geq 0} h_i[a - (h_i + e_j^*) - c]; \ \max_{e_i \geq 0} e_i[a - (e_i + h_j^*) - c] - t_j e_i.$$

- Assuming $e_j^* \leq a - c$ and $h_j^* \leq a - c - t_i$, we have

$$h_i^* = \frac{1}{2}\left(a - e_j^* - c\right) \ ; \ e_i^* = \frac{1}{2}\left(a - h_j^* - c - t_j\right), \quad i = 1, 2.$$

- Solving, we have

$$h_i^* = \frac{a - c - t_i}{3} \ ; \ e_i^* = \frac{a - c - 2t_j}{3}, \quad i = 1, 2.$$

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Tariffs and Imperfect Competition Game: 1st stage

- In the first stage, government i payoff is:

$$W_i(t_i, t_j, h_1^*, e_1^*, h_2^*, e_2^*) = W_i(t_i, t_j)$$

since h_i^* (e_i^*) is a function of t_i (t_j).

- If (t_1^*, t_2^*) is a NE, each government solves:

$$\max_{t_i \geq 0} W_i(t_i, t_j^*).$$

- Solving the optimization, we have $t_i^* = \frac{a-c}{3}$, for $i = 1, 2$. which is a *dominant strategy* for each government.
- Substitute t_i^*, we have

$$h_i^* = \frac{4(a-c)}{9} \; ; \; e_i^* = \frac{a-c}{9}, \text{ for } i = 1, 2.$$

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Comment on Tariffs and Imperfect Competition Game

- In the subgame-perfect outcome, the aggregate quantity on each market is $5(a-c)/9$.
- But if two governments cooperate, they seek socially optimal point and they solve the following optimization problem :

$$\max_{t_1, t_2 \geq 0} W_1(t_1, t_2) + W_2(t_1, t_2)$$

  The solution is $t_1^* = t_2^* = 0$ (no tariff) and the aggregate quantity is $2(a-c)/3$.
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Theory: Two-stage Repeated Game

Prisoners' Dilemma

- Two players play this simultaneously-move game twice.
- Observing the outcome of the first play before the second play begins.

	L_2	R_2
L_1	1,1	5,0
R_1	0,5	4,4

- Payoff for the entire game is the sum of the two stages payoffs.
- Analyzing the 1st stage of the game by taking into account that the outcome of the game remaining in the 2nd stage will be the NE of that game, or (L_1, L_2) with payoff $(1, 1)$.

Theory: Two-stage Repeated Game

Prisoners' Dilemma

- The players' first-stage game amounts to one-shot game:

	L_2	R_2
L_1	2,2	6,1
R_1	1,6	5,5

- This game also has a unique NE: (L_1, L_2).
- The unique subgame-perfect outcome of the 2-stage game is (L_1, L_2) in the first stage, (L_1, L_2) in the 2nd stage.

Definition and Proposition

Definition

Given a stage game G, let $G(T)$ denote the **finitely repeated game** in which G is played T times, with the outcomes of all preceding plays observed before the next play begins. The payoffs for $G(T)$ are simply the sum of the payoffs from the T stage games.

Proposition

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Another look of the 2-stage repeated game

- Consider the following game will be played twice:

	L_2	M_2	R_2
L_1	1,1	5,0	0,0
M_1	0,5	4,4	0,0
R_1	0,0	0,0	3,3

- The stage game has two pure-strategy NE: (L_1, L_2), (R_1, R_2).
- Since more than one NE, players anticipate the different first-stage outcomes will be followed by different stage-game equilibria in the 2nd stage.
- Suppose players anticipate (R_1, R_2) will be the 2nd-stage outcome if the 1st-stage outcome is (M_1, M_2), but (L_1, L_2) will be the 2nd-stage outcome if any of the eight other first-stage outcomes occurs.

Another look of the 2-stage repeated game

- Players' 1st-stage action amounts to the one-shot game:

	L_2	M_2	R_2
L_1	2,2	6,1	1,1
M_1	1,6	7,7	1,1
R_1	1,1	1,1	4,4

- There are three pure-strategy NE: $(L_1, L_2), (M_1, M_2), (R_1, R_2)$.
- The NE (L_1, L_2) corresponds to the subgame-perfect outcome $((L_1, L_2), (L_1, L_2))$ (concatenate 2 NE).
- The NE (R_1, R_2) corresponds to the subgame-perfect outcome $((R_1, R_2), (L_1, L_2))$ (concatenate 2 NE).
- The NE (M_1, M_2) corresponds to the subgame-perfect outcome $((M_1, M_2), (R_1, R_2))$.

Another look of the 2-stage repeated game

Observation

If $G = \{A_1, \ldots, A_n; u_1, \ldots, u_n\}$ is a static game of complete information with multiple NE, then there may be subgame-perfect outcomes of the repeated game $G(T)$ in which for any $t < T$, the outcome in stage t is not a NE of G.

This implies that **credible** promises about the future behavior can influence current behavior

Stronger Result

In infinitely repeated games: even if the stage game has a unique NE, there may be subgame-perfect outcomes of the infinitely repeated games in which no stage's outcome is a NE of G.

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Theory of Infinitely Repeated Game

Let say that we have a static game with complete information, G, and it is repeated infinitely, with the outcomes of all previous stages observed before the current stage begins.

If the payoff is the sum of payoffs at each stage, we have a problem ! To overcome, we have:

Definition (Present Value)

Given the discount factor δ, the present value of the infinite sequence of payoffs $\pi_1, \pi_2, \ldots$ is:

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \ldots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t.$$

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Example an Infinitely Repeated Game

- Consider the following stage game G:

	Player 2: L_2	Player 2: R_2
Player 1: L_1	1,1	5,0
Player 1: R_1	0,5	4,4

- NE of G is (L_1, L_2), but it is **much better** to play (R_1, R_2).
- Player i's strategy is: Play R_i in the 1st stage. In the t^{th} stage, if the outcome of all $t-1$ stages has been (R_1, R_2), then play R_i; otherwise, play L_i.
- This is an example of *trigger strategy*, player i cooperates until someone fails to cooperate, which triggers a switch to non-cooperation forever after.
- Is (R_1, R_2) the NE of this infinitely repeated game?

Proof for Nash Equilibrium when $\delta \geq 1/4$

- Assume player i has adopted the trigger strategy, and show that when δ is close enough to one, player j's best response is to adopt the same trigger strategy.
- Since player i will play L_i forever once the stage outcome differs from (R_1, R_2), player j's best response is to play L_j forever once there is a switch.
- For strategy before the switch, playing L_j will yield a present value

$$5 + \delta * 1 + \delta^2 * 1 + \cdots = 5 + \frac{\delta}{1 + \delta}.$$

- Playing R_j will yield a present value of V where

$$V = 4 + \delta V, \text{ or } V = 4/(1 - \delta).$$

- Player j will choose R_j iff $\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta}$. This is only true if $\delta \geq 1/4$. Trigger strategy is the NE.

More Definitions

Definition (Infinitely Repeated Game)

Given a stage game G, let $G(\infty, \delta)$ denote the infinitely repeated game in which G is repeated forever and the players share the discount factor δ. For each t, the outcomes of the $t-1$ preceding plays of the stage game are observed before the t^{th} stage begins. Each player's payoff in $G(\infty, \delta)$ is the **present value** of the player's payoffs from the infinite sequence of stage games.

More Definitions

Definition (Strategy)

In the finitely repeated game $G(T)$ or the infinitely repeated game $G(\infty, \delta)$, a player's **strategy** specifies the action the player will take in each stage, for each possible history of play through the previous stage.

- Example, previous G has four possible 1st-stage outcomes: $(L_1, L_2), (L_1, R_2), (R_1, L_2), (R_1, R_2)$.
- The player's **strategy** consists of **five** instruction (v, w, x, y, z) where v is the 1st-stage action, the rest are the 2nd-stage actions to be taken following the 4 possible 1st-stage outcomes.
- (1) (b, c, c, c, c) means play b in the 1st-stage, play c no matter what happens in the first. (2) (b, c, c, c, b) means, play b in the 1st-stage, play c in the 2nd-stage unless the 1st-stage outcome was (R_1, R_2), then play b.

Definition on Subgame and Subgame-perfect NE

Definition (Subgame)

In the finitely repeated game $G(T)$, a **subgame** beginning at stage $t+1$ is the repeated game in which G is play $T-t$ times, or $G(T-t)$. There are "*many*" subgames that begin at stage $t+1$, one for each of the possible histories of play through stage t. In the infinitely repeated game $G(\infty, \delta)$, each subgame beginning at stage $t+1$ is identical to the original game $G(\infty, \delta)$. There are as many subgames beginning at stage $t+1$ of $G(\infty, \delta)$ as there are possible histories of play through t.

Definition (Subgame-perfect)

A NE is subgame-perfect if the players' strategies constitute a NE in every subgame.

The trigger-strategy is subgame-perfect NE

- We must show that the trigger strategies constitute a NE on **every subgame** of the infinitely repeated game.
- Note that every subgame of an infinitely repeated game is identical to the game as a whole.
- These subgames can be grouped into two classes: (i) subgames in which all the outcomes of earlier stages are (R_1, R_2), (ii) subgames in which the outcome of at least one earlier stage differs from (R_1, R_2).
- For (i), adopting the trigger strategy, which was shown as a NE of the game.
- For (ii), players repeat the stage-game equilibrium (L_1, L_2), which is also a NE of the game.

Definitions

Definition (Feasible Payoff)

The payoff $(x_1, \ldots, x_n)$ is feasible in the stage game G if there is a **convex combination** of the pure-strategy payoffs of G.

Definition (Average Payoff)

Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\pi_1, \pi_2, \ldots$ is

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \pi_t,$$

The last game has a present value of $4/(1 - \delta)$ but an average payoff of 4.

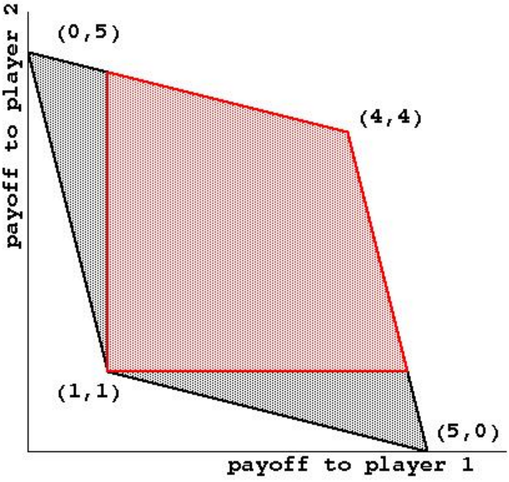
Friedman's Theorem

Friedman's Theorem (or Folk's theorem)

Let G be a finite, static game of complete information. Let $(e_1, \ldots, e_n)$ denote the payoffs from a NE of G, and let $(x_1, \ldots, x_n)$ denote any other feasible payoffs from G. If $x_i > e_i$ $\forall i$ and if δ is sufficiently close to one, then there exits a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, \ldots, x_n)$ as the average payoff.

Folk's Theorem Illustration

	L_2	R_2
L_1	1,1	5,0
R_1	0,5	4,4



Outline

1. Dynamic Game of Complete Information
   - Dynamic Games of Complete & Perfect Information
   - Two-Stage Dynamic Games of Complete but Imperfect Information
   - Repeated Games

2. **Static Games of Incomplete Information**
   - Theory: Static Bayesian Games and Bayesian NE

An Example: Cournot Competition under Asymmetric Information

- Consider a Cournot duopoly model with inverse demand given by $P(Q) = a - Q$ where $Q = q_1 + q_2$ is the aggregate quantity.
- Firm 1's cost function is $C_1(q_1) = cq_1$.
- Firm 2's cost function is $C_2(q_2) = c_H q_2$ with probability θ and $C_2(q_2) = c_L q_2$ with $(1 - \theta)$, where $c_L < c_H$.
- Information is asymmetric: firm 2 knows its cost function and firm 1's, but firm 1 knows its cost function and only that firm 2's marginal cost c_H with θ and c_L with $(1 - \theta)$.
- This is a game with *incomplete and asymmetric information*.

Cournot game with incomplete information

- Let $q_2^*(c_H)$ and $q_2^*(c_L)$ be firm 2's quantity choices, q_1^* be firm 1's single quantity choice.
- If firm 2's cost is high, it will choose $q_2^*(c_H)$ and to solve

$$\max_{q_2} \left[(a - q_1^* - q_2) - c_H\right] q_2.$$

Similarly, if cost is low, $q_2^*(c_L)$ will solve

$$\max_{q_2} \left[(a - q_1^* - q_2) - c_L\right] q_2.$$

- Firm 1 chooses q_1^* to solve

$$\max_{q_1} \theta \left[(a - q_1 - q_2^*(c_H)) - c\right] q_1 + (1 - \theta) \left[(a - q_1 - q_2^*(c_L)) - c\right] q_1.$$

Cournot game with incomplete information

- Solving these optimization problems, we have

$$\begin{aligned} q_2^*(c_H) &= \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L), \\ q_2^*(c_L) &= \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L), \\ q_1^* &= \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}. \end{aligned}$$

- For Cournot game with *complete information*, $q_i^* = (a - 2c_i + c_j)/3$, for $i = 1, 2$.
- Note that under the Counot game with *incomplete information*, $q_2^*(c_H) > q_2^*$ and $q_2^*(c_L) < q_2^*$.
- **WHY?** Because firm 2 not only tailors its quantity to its cost, but also **anticipate** the response by firm 1.

Definition

Definition (Bayesian Nash Equilibrium)

In the static Bayesian game $G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\}$. The strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a (pure-strategy) Bayesian Nash equilibrium if for each player i and for each i's types t_i in T_i, $s_i^*(t_i)$ solves:

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), \ldots, s_n^*(t_n); t) p_i(t_{-i}) | t_i).$$

Example:

- Consider the following game:

	Pat **(Opera)**	Pat **(Football Game)**
Chris **(Opera)**	2,1	0,0
Chris **(Football Game)**	0,0	1,2

- Two NE under pure strategy, (Opera, Opera), (FG, FG).
- What is the *mixed strategy* that has the NE?

Example:

- Let q (r) be the probability that Pat (Chris) will choose Opera.
- Chris's expected payoff in choosing Opera is $q \times 2 + (1 - q) \times 0 = 2q$, and the expected payoff in choosing FG is $q \times 0 + (1 - q) \times 1 = 1 - q$. So Chris will choose opera iff $q > 1/3$, will choose FG iff $q < 1/3$. If $q = 1/3$, any value of r is the best response by Chris.
- Pat's expected payoff in choosing Opera is $r \times 1 + (1 - r) \times 0 = r$, and the expected payoff in choosing FG is $r \times 0 + (1 - r) \times 2 = 2(1 - r)$. So Pat will choose opera iff $r > 2/3$, will choose FG iff $r < 2/3$. If $r = 2/3$, any value of q is the best response by Pat.
- Then $(q, 1 - q) = (1/3, 2/3)$ for Pat and $(r, 1 - r) = (2/3, 1/3)$ for Chris are the mixed strategy NE.

Example:

- Consider the following static game with incomplete information:

	Pat **(Opera)**	Pat **(FG)**
Chris **(Opera)**	2+t_c,1	0,0
Chris **(FG)**	0,0	1,2 +t_p

  where t_c (t_p) is privately known by Chris (Pat) only. Both t_p and t_c are independent and uniformly distributed in $[0, x]$.

- The **normal form** $G = \{A_c, A_p; T_c, T_p; p_c, p_p; u_c, u_p\}$.
  $A_c = A_p =$\{Opera,FG\}, $T_c = T_p = [0, x]$,
  $p_c(t_p) = p_p(t_c) = 1/x$ for all t_c and t_p.
- What is the pure-strategy Bayesian Nash equilibrium of this game?

Solution

- We'll construct a pure-strategy BNE in which Chris chooses opera if $t_c > c$ and chooses FG otherwise, while Pat chooses FG if $t_p > p$ and chooses opera otherwise.
- In such an equilibrium, Chris chooses opera with probability $(x - c)/x$ while Pat chooses FG with probability $(x - p)/x$.
- Note that when the incomplete information disappears (i.e., as $x \to 0$), the BNE should approach the mixed-strategy NE, or $(x - c)/x$ and $(x - p)/x$ will approach $2/3$ as x approaches zero.

Solution: continue

- For a given value of x, we will determine values of c and p such that these strategies are a BNE.
- Given Pat's strategy, Chris's expected payoff of opera & FG:

$$\begin{aligned}\frac{p}{x} \times (2+t_c) + \left(1-\frac{p}{x}\right) \times 0 &= \frac{p}{x}(2+t_c), \\ \frac{p}{x} \times 0 + \left(1-\frac{p}{x}\right) \times 1 &= 1-\frac{p}{x}.\end{aligned}$$

  Thus, Chris chooses opera iff $t_c \geq \frac{x}{p} - 3 = c$.
- Gvien Chris's strategy, Pat's expected payoff of opera & FG:

$$\begin{aligned}\left(1-\frac{c}{x}\right) \times 1 + \frac{c}{x} \times 0 &= 1-\frac{c}{x}, \\ \left(1-\frac{c}{x}\right) \times 0 + \frac{c}{x} \times (2+t_p) &= \frac{c}{x}(2+t_p),\end{aligned}$$

  Thus, Pat chooses FG iff $t_p \geq \frac{x}{c} - 3 = p$.

Solution: continue

- Equating t_c and t_p, we have two equations: $p = c$ and $p^2 + 3p - x = 0$.
- Solving the quadratic equation shows that:

$$\text{Prob[Chris chooses Opera]} = \frac{x-c}{x} = 1 - \frac{-3+\sqrt{9+4x}}{2x},$$
$$\text{Prob[Pat chooses FG]} = \frac{x-p}{x} = 1 - \frac{-3+\sqrt{9+4x}}{2x}.$$

Both approach $2/3$ as x approaches zero.

- Thus, the incomplete information disappears, the player's behavior in this pure-strategy BNE of the incomplete-information game approaches the mixed-strategy NE of the game with complete information.