CSC5420
Phase-type Systems

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Method of Stages: Erlangian Distribution E_r

more general than birth-death

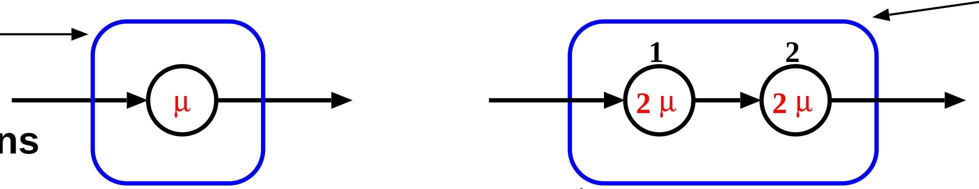
- **Let service time density function**

Recall: expo.

$$b(x) = \mu e^{-\mu x} \qquad x \geq 0$$

$$B^*(s) = \frac{\mu}{s+\mu}; E[\tilde{x}] = \frac{1}{\mu}; \sigma_b^2 = \frac{1}{\mu^2}$$

Decompose service time onto structured expo. distributions



one customer in this box at a time

service facility

each ◯

$$B^*(s) = \int_0^\infty e^{-sx} b(x)dx$$

$$H^*(s) = \int_0^\infty e^{-sy} b(y)dy$$

$$h(y) = 2\mu e^{-2\mu y} \qquad y \geq 0; E[\tilde{y}] = \frac{1}{2\mu}; \sigma_h^2 = \left(\frac{1}{2\mu}\right)^2$$

$$x = y + y$$

$$B^*(s) = [H^*(s)]^2$$

$$H^*(s) = \frac{2\mu}{s+2\mu}$$

$$B^*(s) = \left(\frac{2\mu}{s+2\mu}\right)^2$$

Erlangian Distribution E_r (Cont...)

From transform table

$$X^*(s) = \left(\frac{\lambda}{s+\lambda}\right)^k \Rightarrow f_X(x) = \frac{\lambda(\lambda x)^{k-1}}{(k-1)!}e^{-\lambda x} \quad x \geq 0; k \geq 1$$

Therefore:

$$b(x) = 2\mu(2\mu x)e^{-2\mu x} \quad x \geq 0$$

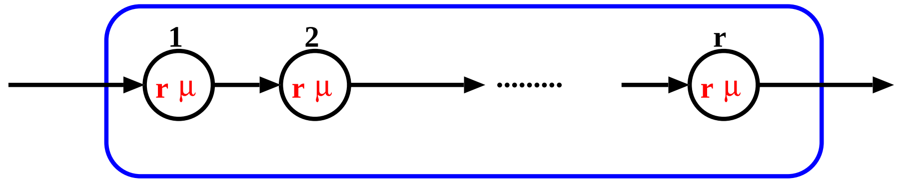
$$E[x] = E[y] + E[y] = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu}$$

$$\sigma_b^2 = \sigma_n^2 + \sigma_n^2 = \frac{2}{(2\mu)^2} = \frac{1}{2\mu^2}$$

ind. r.v. **HW: can verify through transform or density func.**

mean same as expo.; var. half of expo.

E_r: r-stage Erlangian distribution



$$h(y) = r\mu e^{-r\mu y} \qquad y \geq 0$$

$$E[y] = \frac{1}{r\mu} \quad ; \quad \sigma_h^2 = \frac{1}{(r\mu)^2} \quad ; \quad E[x] = r\frac{1}{r\mu} = \frac{1}{\mu}$$

$$\sigma_x^2 = r\left(\frac{1}{r\mu}\right)^2 = \frac{1}{r\mu^2} \quad ; \quad C_b = \frac{1}{\sqrt{r}}$$ **coeff. of var.**

$$B^*(s) = \left[\frac{r\mu}{s+r\mu}\right]^r \Rightarrow b(x) = \frac{r\mu(r\mu x)^{r-1}}{(r-1)!}e^{-r\mu x} \qquad x \geq 0$$

Erlang distribution

$M/E_r/1$ System

$$a(t) = \lambda e^{-\lambda t}$$
$$b(x) = \frac{r\mu(r\mu x)^{r-1}}{(r-1)!}e^{-r\mu x} \quad x \geq 0$$

# of stages to do for customer in service

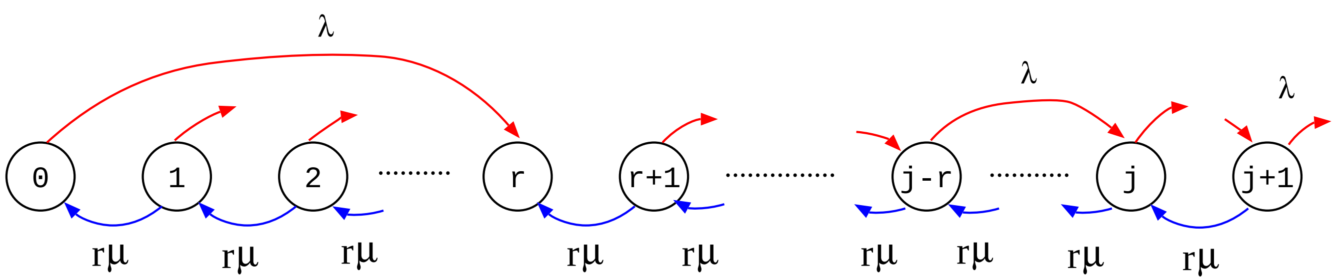
- **State description: $[k, s_l]$ transform to $[s]$ where s is the total number of stages yet to be completed by all customers.**
- **If the system has k customers and when the i^{th} stage of service contains the customers.**

$$\begin{aligned} j &= \text{number of stages left in the total system} \\ &= (k-1)r + (r-i+1) = rk - i + 1 \end{aligned}$$

- **Let P_j be the probability of j stages of work in the system. Since $j=rk-i+1$, we have:**

$$p_k = \text{Prob}[k \text{ customers}] = \sum_{j=(k-1)r+1}^{rk} P_j \quad k = 1, 2, \cdots$$

$M/E_r/1$ System (Cont...)



- **Let $P_j=0$ for $j < 0$.**

$$\begin{aligned}\lambda P_0 &= r\mu P_1 \\ (\lambda + r\mu)P_j &= \lambda P_{j-r} + r\mu P_{j+1} \qquad j = 1, 2, \cdots\end{aligned}$$

- **Define $P(z) = \sum_{j=0}^{\infty} P_j z^j$**

$$\begin{aligned}\sum_{j=1}^{\infty}(\lambda + r\mu)P_j z^j &= \sum_{j=1}^{\infty}\lambda P_{j-r} z^j + \sum_{j=1}^{\infty} r\mu P_{j+1} z^j \\ (\lambda + r\mu)\left[P(z) - P_0\right] &= \lambda z^r \left[P(z)\right] + \frac{r\mu}{z}\left[P(z) - P_0 - P_1 z\right]\end{aligned}$$

$M/E_r/1$ System (Cont...)

$$P(z) = \frac{P_0\left[\lambda + r\mu - \left(\frac{r\mu}{z}\right)\right] + r\mu P_1}{\lambda + r\mu - \lambda z^r - \left(\frac{r\mu}{z}\right)} = \frac{P_0 r\mu(1-\frac{1}{z})}{\lambda + r\mu - \lambda z^r - \left(\frac{r\mu}{z}\right)}$$

- **Since $P(1)=1$, therefore, using L' Hospital rule, we have**

$$1 = \frac{r\mu P_0}{r\mu - \lambda r}, \text{therefore, } P_0 = \frac{r\mu - \lambda r}{r\mu} = 1 - \frac{\lambda}{\mu}$$

$p_0 = P_0 \Rightarrow$ so still same notion of util.

- **Define $\rho = \lambda/\mu$, we have**

$$P(z) = \frac{r\mu(1-\rho)(1-z)}{r\mu + \lambda z^{r+1} - (\lambda + r\mu)z}$$

$\Rightarrow$ try for $r=1$ to match $M/M/1$ solution

M/E$_r$*/1* System (Cont...)

- For general r, look at denominator, there are $(r+1)$ zeros. Unity is one of them; we have

$$(1-z)[r\mu - \lambda(z + z^2 + \cdots + z^r)]$$

Therefore, we have r zeros which are $z_1, z_2, \ldots, z_r$. We can arrange them to be

$$r\mu\left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right)\cdots\left(1-\frac{z}{z_r}\right)$$

We have

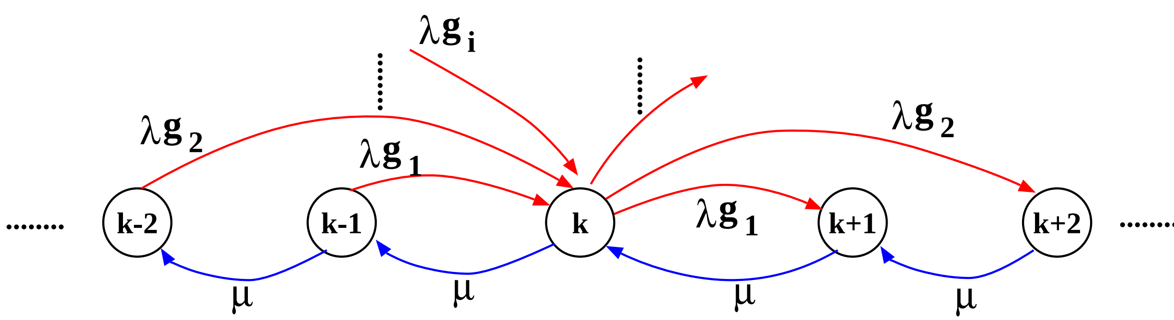
$$P(z) = (1-\rho)\sum_{i=1}^{r}\frac{A_i}{1-\frac{z}{z_i}}$$

Need to resolve this by partial fraction expansion.

- For $E_r/M/1$ system, derive it at home.

Bulk Arrival Systems

- **Let g_i be the probability that the bulk size is i, for $i > 0$.**



$$
\begin{aligned}
\lambda P_0 &= \mu P_1 \\
(\lambda+\mu)P_k &= \mu P_{k+1} + \sum_{i=0}^{k-1} \lambda g_{k-i} P_i \qquad k \geq 1 \\
(\lambda+\mu)\sum_{k=1}^{\infty} P_k z^k &= \mu \sum_{k=1}^{\infty} P_{k+1} z^k + \lambda \sum_{k=1}^{\infty}\sum_{i=0}^{k-1} g_{k-i} P_i z^K
\end{aligned}
$$

Bulk Arrival Systems (Cont...)

$$(\lambda + \mu)\left[P(z) - P_0\right] = \frac{\mu}{z}\left[P(z) - P_0 - P_1 z\right] + \lambda P(z)G(z)$$

$$P(z) = \frac{\mu P_0(1-z)}{\mu(1-z) - \lambda z[1-G(z)]}$$

transform for distri. of bulk sizes

$$G(z) = \sum_{k=1}^{\infty} g_k z^k$$

- **Using $P(1) = 1$ and L' Hospital rule**

$$P(z) = \frac{\mu(1-\rho)(1-z)}{\mu(1-z) - \lambda z[1-G(z)]}$$

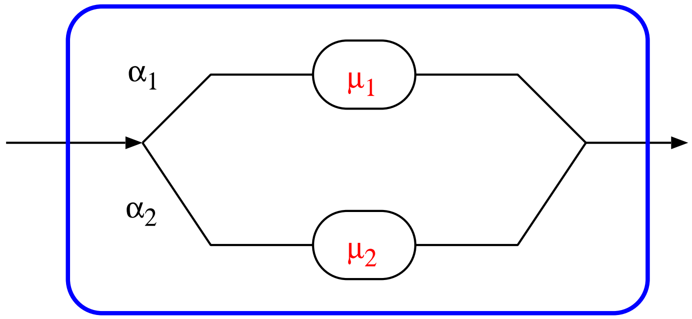
where

$$\rho = \frac{\lambda G'(1)}{\mu}$$

avg. bulk size

- **For bulk service system, try it at home.**

Parallel System



$$b(x) = \alpha_1 \mu_1 e^{-\mu_1 x} + \alpha_2 \mu_2 e^{-\mu_2 x} \quad x \geq 0$$

$$B^*(s) = \alpha_1 \left(\frac{\mu_1}{s+\mu_1} \right) + \alpha_2 \left(\frac{\mu_2}{s+\mu_2} \right)$$

- **In general, if we have R parallel stages (hyper-exponential):**

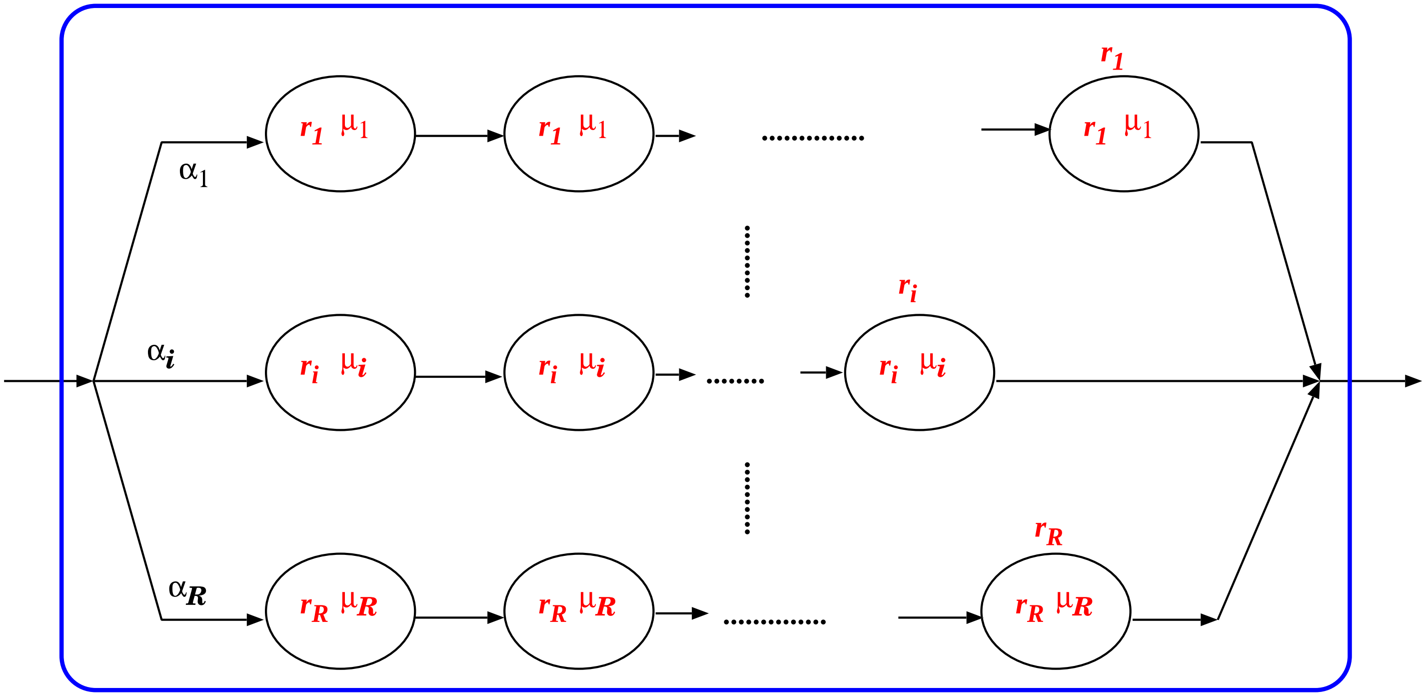
$$B^*(s) = \sum_{i=1}^{R} \alpha_i \left(\frac{\mu_i}{s+\mu_i} \right)$$

$$b(x) = \sum_{i=1}^{R} \alpha_i \mu_i e^{-\mu_i x} \quad x \geq 0$$

Parallel System (Cont...)

$$
\begin{aligned}
\bar{x} &= \sum_{i=1}^{R} \alpha_i \left(\frac{1}{\mu_i} \right) \\
\bar{x^2} &= \sum_{i=1}^{R} \alpha_i \left(\frac{1}{\mu_i^2} \right) \\
C_b^2 &= \frac{\sigma_b^2}{(\bar{x})^2} = \frac{\bar{x^2} - (\bar{x})^2}{(\bar{x})^2} \quad \Rightarrow \quad \boxed{C_b^2 \geq 1}
\end{aligned}
$$

Series and Parallel System



$$b(x) = \sum_{i=1}^{R} \alpha_i \frac{r_i \mu_i (r_i \mu_i x)^{r_i - 1}}{(r_i - 1)!} e^{-r_i \mu_i x} \qquad x \geq 0$$

$$B^*(s) = \sum_{i=1}^{R} \alpha_i \left(\frac{r_i \mu_i}{s + r_i \mu_i} \right)^{r_i}$$

Series and Parallel System (Cont...)

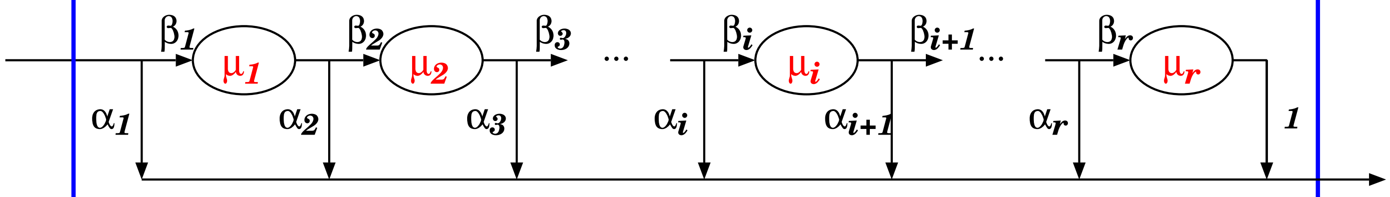
Generalization of series-parallel server

- **remove restrition that each stage within the same branch has same service rate**
  - $\Rightarrow \mu_{ij} \Rightarrow$ **service rate of i^{th} branch, j^{th} stage**

$$\Rightarrow B^*(s) = \sum_{j=1}^{R} \alpha_i \prod_{i=1}^{r_i} \left(\frac{\mu_{ij}}{s + \mu_{ij}} \right)$$

Series and Parallel System (Cont...)

Another way to create series-parallel effect



- Before entering i^{th} stage, indepedent choice
  - with prob. β_i proceed to i^{th} expo. service stage
  - with prob. α_i depart

    $$\Rightarrow \beta_i + \alpha_i = 1 \qquad i = 1, 2, \cdots, r$$
  - after r^{th} stage depart with prob. *1*

$$\Rightarrow B^*(s) = \alpha_1 + \sum_{i=1}^{r} \beta_1\beta_2 \cdots \beta_i \alpha_{i+1} \prod_{j=1}^{i} \left(\frac{\mu_j}{s + \mu_j} \right) \text{ where } \alpha_{r+1} = 1$$

(Cox showed that more complex transitions in above do not give more general service distributions.)

(If permit complex values for $r_i\mu_i$, then can syntherize any rational func. of s_i, can approx. non-rat. func. can be approx. arb. close.)