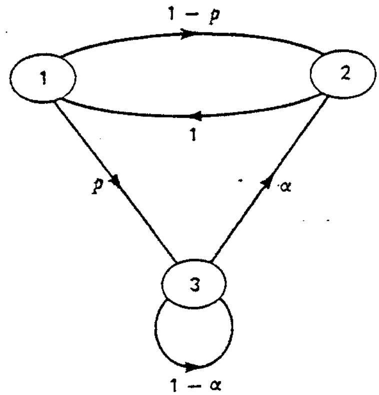
2.4. Find the pdf for the smallest of K independent random variables, each of which is exponentially distributed with parameter λ.

2.5. Consider the homogeneous Markov chain whose state diagram is

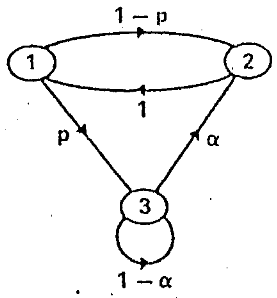


(a) Find P, the probability transition matrix.
(b) Under what conditions (if any) will the chain be irreducible and aperiodic?

(c) Solve for the equilibrium probability vector π.
(d) What is the mean recurrence time for state E_2?
(e) For which values of α and p will we have $\pi_1 = \pi_2 = \pi_3$? (Give a physical interpretation of this case.)

PROBLEM 2.5

Consider the homogeneous Markov chain whose state diagram is



(a) Find $\mathbf{P}$, the probability transition matrix.
(b) Under what conditions (if any) will the chain be irreducible and aperiodic?
(c) Solve for the equilibrium probability vector π.
(d) What is the mean recurrence time for state 2?
(e) For which values of α and p will we have $\pi_1 = \pi_2 = \pi_3$? (Give a physical interpretation of this case.)

SOLUTION

(a) We have

$$\mathbf{P} = \begin{bmatrix} 0 & 1-p & p \\ 1 & 0 & 0 \\ 0 & \alpha & 1-\alpha \end{bmatrix}$$

(b) Irreducible and aperiodic for all $0 < p \leq 1$ and $0 < \alpha \leq 1$ except $\alpha = p = 1$.
(c) From $\pi = \pi\mathbf{P}$ [$\pi = (\pi_1, \pi_2, \pi_3)$] we obtain only two independent equations, namely,

$$\pi_1 = \pi_2$$

$$\pi_2 = (1-p)\pi_1 + \alpha\pi_3$$

Using the conservation of probability, we also have $\pi_1 + \pi_2 + \pi_3 = 1$. Thus

$$\pi_1 = \pi_2 = \frac{\alpha}{p + 2\alpha}$$

$$\pi_3 = \frac{p}{p + 2\alpha}$$

(d) We find that

$$\mu_2 = \frac{1}{\pi_2} = \frac{p + 2\alpha}{\alpha} = 2 + \frac{p}{\alpha}$$

(e) We need

$$\alpha = p$$

Interpretation: Since each visit to state 1 is followed by exactly one visit to state 2 and vice versa for all p and α, we have $\pi_1 = \pi_2$ always. Also p ($= \alpha$) of the time we go from state 1 to state 3 and the average number of steps (or mean time) spent in state 3 per visit is $1/[1 - (1-\alpha)] = 1/\alpha$. Thus $\alpha \cdot (1/\alpha) = 1$ is the average number of visits to state 3 per visit to state 1. □

PROBLEM 2.4

Find the pdf for the smallest of K independent random variables, each of which is exponentially distributed with parameter λ.

SOLUTION

Let the K random variables be $X_1, X_2, \ldots, X_K$. The random variable of interest is $Y = \min(X_1, X_2, \ldots, X_K)$.

$$\begin{aligned} P[Y > y] &= P[X_1 > y, \ldots, X_K > y] \\ &= P[X_1 > y] \cdots P[X_K > y] \qquad (X_i \text{ are independent}) \\ &= e^{-\lambda y} \cdots e^{-\lambda y} = e^{-K\lambda y} \end{aligned}$$

Thus Y is exponential with parameter $K\lambda$; that is,

$$P[Y \leq y] = 1 - e^{-K\lambda y}$$ □

EXERCISES

3.1. Consider a pure Markovian queueing system in which

$$\lambda_k = \begin{cases} \lambda & 0 \le k \le K \\ 2\lambda & K < k \end{cases}$$

$$\mu_k = \mu \qquad k = 1, 2, \ldots$$

(a) Find the equilibrium probabilities p_k for the number in the system.

(b) What relationship must exist among the parameters of the problem in order that the system be stable and, therefore, that this equilibrium solution in fact be reached? Interpret this answer in terms of the possible dynamics of the system.

3.2. Consider a Markovian queueing system in which

$$\lambda_k = \alpha^k \lambda \qquad k \ge 0,\ 0 \le \alpha < 1$$

$$\mu_k = \mu \qquad k \ge 1$$

(a) Find the equilibrium probability p_k of having k customers in the system. Express your answer in terms of p_0.

(b) Give an expression for p_0.

3.3. Consider an M/M/2 queueing system where the average arrival rate is λ customers per second and the average service time is $1/\mu$ sec, where $\lambda < 2\mu$.

(a) Find the differential equations that govern the time-dependent probabilities $P_k(t)$.

(b) Find the equilibrium probabilities

$$p_k = \lim_{t \to \infty} P_k(t)$$

3.4. Consider an M/M/1 system with parameters λ, μ in which customers are impatient. Specifically, upon arrival, customers estimate their queueing time w and then join the queue with probability $e^{-\alpha w}$ (or leave with probability $1 - e^{-\alpha w}$). The estimate is $w = k/\mu$ when the new arrival finds k in the system. Assume $0 \le \alpha$.

(a) In terms of p_0, find the equilibrium probabilities p_k of finding k in the system. Give an expression for p_0 in terms of the system parameters.

(b) For $0 < \alpha$, $0 < \mu$ under what conditions will the equilibrium solution hold?

(c) For $\alpha \to \infty$, find p_k explicitly and find the average number in the system.

3.5. Consider a birth–death system with the following birth and death coefficients:

$$\lambda_k = (k + 2)\lambda \qquad k = 0, 1, 2, \ldots$$

$$\mu_k = k\mu \qquad k = 1, 2, 3, \ldots$$

All other coefficients are zero.

(a) Solve for p_k. Be sure to express your answer explicitly in terms of λ, k, and μ only.

(b) Find the average number of customers in the system.

3.6. Consider a birth–death process with the following coefficients:

$$\lambda_k = \alpha k(K_2 - k) \qquad k = K_1, K_1 + 1, \ldots, K_2$$

$$\mu_k = \beta k(k - K_1) \qquad k = K_1, K_1 + 1, \ldots K_2$$

where $K_1 \le K_2$ and where these coefficients are zero outside the range $K_1 \le k \le K_2$. Solve for p_k (assuming that the system initially contains $K_1 \le k \le K_2$ customers).

PROBLEM 3.1

Consider a birth–death queueing system in which

$$\lambda_k = \begin{cases} \lambda & 0 \le k \le K \\ 2\lambda & K < k \end{cases}$$

$$\mu_k = \mu \qquad k = 1, 2, \ldots$$

(a) Find the equilibrium probabilities p_k for the number in the system.

(b) What relationship must exist among the parameters of the problem in order that a solution exist? Interpret this answer in terms of the possible dynamics of the system.

SOLUTION

(a) Case (1): $0 \le k \le K+1$

Equation (1.63) gives

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^k$$

Case (2): $k > K+1$

Equation (1.63) gives

$$p_k = p_0 \prod_{i=0}^{K} \frac{\lambda}{\mu} \prod_{i=K+1}^{k-1} \frac{2\lambda}{\mu}$$

$$p_k = p_0 \left[\left(\frac{\lambda}{\mu}\right)^{K+1} \left(\frac{2\lambda}{\mu}\right)^{k-K-1}\right] = p_0 \frac{1}{2^{K+1}} \left(\frac{2\lambda}{\mu}\right)^k$$

Using the conservation of probability relation, namely, $\sum_{k=0}^{\infty} p_k = 1$, we solve for p_0 as follows:

$$1 = p_0 \left[\sum_{k=0}^{K+1} \left(\frac{\lambda}{\mu}\right)^k + \sum_{k=K+2}^{\infty} \frac{1}{2^{K+1}} \left(\frac{2\lambda}{\mu}\right)^k\right]$$

$$= p_0 \left[\frac{1-\left(\frac{\lambda}{\mu}\right)^{K+2}}{1-\frac{\lambda}{\mu}} + \frac{1}{2^{K+1}} \left(\frac{2\lambda}{\mu}\right)^{K+2} \frac{1}{1-\frac{2\lambda}{\mu}}\right]$$

Thus

$$p_0 = \frac{\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{2\lambda}{\mu}\right)}{1-\frac{2\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^{K+2}}$$

and

$$p_k = \begin{cases} \dfrac{\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{2\lambda}{\mu}\right)}{1-\frac{2\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^{K+2}} \left(\frac{\lambda}{\mu}\right)^k & 0 \le k \le K+1 \\[2ex] \dfrac{\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{2\lambda}{\mu}\right)}{1-\frac{2\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^{K+2}} \left(\frac{2\lambda}{\mu}\right)^k \frac{1}{2^{K+1}} & k > K+1 \end{cases}$$

(b) We must have $(2\lambda/\mu) < 1$ or $2\lambda < \mu$ to ensure that p_k does not produce a divergent series. We observe that if the system goes unstable, then the number in system will exceed K with probability one. Thus the relevant birth

PROBLEM 3.2

Consider a birth–death queueing system in which

$$\lambda_k = \alpha^k \lambda \qquad k \geq 0, 0 \leq \alpha < 1$$
$$\mu_k = \mu \qquad k \geq 1$$

(a) Find the equilibrium probability p_k of having k customers in the system. Express your answer in terms of p_0.
(b) Give an expression for p_0.

SOLUTION

(a) From Eq. (1.63) we have

$$p_k = p_0 \prod_{i=0}^{k-1} \alpha^i \left(\frac{\lambda}{\mu}\right) = p_0 \left(\frac{\lambda}{\mu}\right)^k \alpha^{\sum_{i=0}^{k-1} i}$$

$$p_k = p_0 \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{(k-1)k}{2}}$$

Therefore

$$p_k = p_0 \left[\frac{\lambda \alpha^{(k-1)/2}}{\mu}\right]^k$$

(b) Using conservation of probability, we find

$$\sum_{k=0}^{\infty} p_k = 1 = p_0 \sum_{k=0}^{\infty} \left[\frac{\lambda \alpha^{(k-1)/2}}{\mu}\right]^k$$

So

$$p_0 = \frac{1}{\sum_{k=0}^{\infty} \left[\frac{\lambda \alpha^{(k-1)/2}}{\mu}\right]^k}$$

Note for $0 \leq \alpha < 1$, this system is *always* stable. □

PROBLEM 3.4

Consider an M/M/1 system with parameters λ, μ in which customers are impatient. Specifically, upon arrival, customers estimate their queueing time w and then join the queue with probability $e^{-\alpha w}$ (or leave with probability $1 - e^{-\alpha w}$). The estimate is $w = k/\mu$ when the new arrival finds k in the system. Assume $0 \leq \alpha$.

(a) In terms of p_0, find the equilibrium probabilities p_k of finding k in the system. Give an expression for p_0 in terms of the system parameters.

(b) Under what conditions will the equilibrium solution hold (i.e., when will $p_0 > 0$)?

(c) For $\alpha \to \infty$, find p_k explicitly and find the average number in the system.

SOLUTION

The birth and death coefficients are

$$\lambda_k = \lambda e^{-\frac{\alpha k}{\mu}}, \; \mu_k = \mu$$

(a) Equation (1.63) gives

$$p_k = p_0 \prod_{i=0}^{k-1} \frac{\lambda e^{-\frac{\alpha i}{\mu}}}{\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^k e^{-\frac{\alpha}{\mu}\sum_{i=0}^{k-1} i}$$

$$p_k = p_0 \left(\frac{\lambda}{\mu}\right)^k e^{-\frac{\alpha k(k-1)}{2\mu}}$$

$$\sum_{k=0}^{\infty} p_k = 1 = p_0 \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k e^{-\frac{\alpha k(k-1)}{2\mu}}$$

$$p_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k e^{-\frac{\alpha k(k-1)}{2\mu}}}$$

(b) The denominator for p_0 converges if either $0 < \alpha$ and $0 < \mu$ or if $\alpha = 0$ and $\lambda < \mu$.

(c) For $\alpha \to \infty$, $p_k \to 0$ for $k \geq 2$. Thus we only move between the two states 0 and 1 with $\lambda_0 = \lambda$, $\lambda_1 = 0$ and $\mu_1 = \mu$. Thus solving for p_0 and p_1 (see also Problem 2.11 for $P_k(t)$) gives

$$p_0 = \frac{\mu}{\lambda + \mu}$$

$$p_1 = \frac{\lambda}{\lambda + \mu}$$

$$\overline{N} = 0 \cdot p_0 + 1 \cdot p_1 = \frac{\lambda}{\lambda + \mu}$$

□

PROBLEM 3.5

Consider a birth–death system with the following birth and death coefficients:

$$\lambda_k = (k+2)\lambda \qquad k = 0, 1, 2, \ldots$$

$$\mu_k = k\mu \qquad k = 1, 2, 3, \ldots$$

All other coefficients are zero.

(a) Solve for p_k. Be sure to express your answer explicitly in terms of λ, k, and μ only.

(b) Find the average number of customers in the system.

SOLUTION

(a) Equation (1.63) gives

$$p_k = p_0 \left(\frac{\lambda}{\mu}\right)^k \frac{2 \cdot 3 \cdots (k+1)}{1 \cdot 2 \cdots k}$$

$$p_k = p_0 (k+1) \left(\frac{\lambda}{\mu}\right)^k \qquad \text{for } k \geq 0$$

$$1 = \sum_{k=0}^{\infty} p_k = p_0 \sum_{k=0}^{\infty} (k+1) \left(\frac{\lambda}{\mu}\right)^k$$

Here we demonstrate the "differentiation trick" for summing series. Since

$$\sum_{k=0}^{\infty} (k+1) \left(\frac{\lambda}{\mu}\right)^k = \frac{\partial}{\partial\left(\frac{\lambda}{\mu}\right)} \sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{k+1}$$

we have

$$\sum_{k=0}^{\infty} (k+1) \left(\frac{\lambda}{\mu}\right)^k = \frac{\partial}{\partial\left(\frac{\lambda}{\mu}\right)} \left[\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}\right] = \frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2}$$

Thus

$$p_0 = \left(1-\frac{\lambda}{\mu}\right)^2$$

and so

$$p_k = \left(1-\frac{\lambda}{\mu}\right)^2 (k+1) \left(\frac{\lambda}{\mu}\right)^k \qquad k \geq 0$$

(b) We have

$$\overline{N} = \sum_{k=0}^{\infty} k p_k = \sum_{k=1}^{\infty} k \left(1-\frac{\lambda}{\mu}\right)^2 (k+1) \left(\frac{\lambda}{\mu}\right)^k$$

$$= \left(1-\frac{\lambda}{\mu}\right)^2 \frac{\lambda}{\mu} \frac{\partial^2}{\partial\left(\frac{\lambda}{\mu}\right)^2} \left[\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^{k+1}\right]$$

$$\overline{N} = \left(1-\frac{\lambda}{\mu}\right)^2 \frac{\lambda}{\mu} \frac{\partial}{\partial\left(\frac{\lambda}{\mu}\right)} \left[\frac{1}{\left(1-\frac{\lambda}{\mu}\right)^2}\right] = \left(1-\frac{\lambda}{\mu}\right)^2 \frac{\lambda}{\mu} \frac{2}{\left(1-\frac{\lambda}{\mu}\right)^3}$$

$$\overline{N} = \frac{2\left(\frac{\lambda}{\mu}\right)}{1-\frac{\lambda}{\mu}}$$ □

PROBLEM 3.6

Consider a birth–death process with the following coefficients:

$$\lambda_k = \alpha k (K_2 - k) \qquad k = K_1, K_1+1, \ldots, K_2$$

$$\mu_k = \beta k (k - K_1) \qquad k = K_1, K_1+1, \ldots, K_2$$

where $K_1 \leq K_2$ and where these coefficients are zero outside the range $K_1 \leq k \leq K_2$. Solve for p_k (assuming that the system initially contains $K_1 \leq k \leq K_2$ customers).

SOLUTION

Clearly $p_k = 0$ for $k < K_1$, $k > K_2$. Using the obvious translation of Eq. (1.63) in the range $K_1 \leq k \leq K_2$, we get

$$p_k = p_{K_1} \prod_{i=K_1}^{k-1} \frac{\alpha i (K_2 - i)}{\beta (i+1)(i+1-K_1)}$$

$$p_k = p_{K_1} \left(\frac{\alpha}{\beta}\right)^{k-K_1} \frac{K_1(K_1+1)\cdots(k-1)}{(K_1+1)(K_1+2)\cdots k} \cdot \frac{(K_2-K_1)\cdots(K_2-k+1)}{1 \cdot 2 \cdots (k-K_1)}$$

Multiplying the top and bottom of the right-hand expression by $K_1!\,(K_2-k)!$ we find

$$p_k = p_{K_1} \left(\frac{\alpha}{\beta}\right)^{k-K_1} \frac{(k-1)!\,K_1(K_2-K_1)!}{(k-1)!\,k(K_2-k)!\,(k-K_1)!}$$

$$p_k = p_{K_1} \left(\frac{\alpha}{\beta}\right)^{k-K_1} \frac{K_1}{k} \binom{K_2-K_1}{k-K_1} \qquad k = K_1, \ldots, K_2$$

where, by conserving probability, we get

$$p_{K_1} = \frac{1}{\sum_{k=K_1}^{K_2} \left(\frac{\alpha}{\beta}\right)^{k-K_1} \frac{K_1}{k} \binom{K_2-K_1}{k-K_1}}$$ □