① Let $f_n = n 2^n + 4^n \qquad n = 0, 1, 2, \ldots$

Find $F(z) = \sum_{n=0}^{\infty} f_n z^n$

② Let $F(z) = \dfrac{2z - 1}{-3z^2 + 4z - 1}$

Find $f_n \Leftrightarrow F(z) \qquad n = 0, 1, 2, \ldots$

③ Given $G(z) = F(z) + z^2 F(z)$

where $F(z)$ is as given as in problem (2).

Find $g_n \Leftrightarrow G(z) \qquad n = 0, 1, 2, \ldots$

④ Let $f(t) = u_0(t) + t e^{-t} + e^{2t} \qquad t \geq 0$

Find $F^*(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt$

⑤ Let $F^*(s) = \dfrac{s(s+1)}{(s+2)(s+3)}$

Find $f(t) \Leftrightarrow F^*(s) \qquad t \geq 0$

⑥ Let $10 f_n - 7 f_{n-1} + f_{n-2} = \left(\frac{1}{3}\right)^n \qquad n = 2, 3, 4, \ldots$

and $f_0 = 0, \; f_1 = 3$

a) Find $f_n \qquad n = 0, 1, 2, \ldots$ <u>without</u> using transforms

b) Find $f_n \qquad n = 0, 1, 2, \ldots$ using transform

7 Let $$\frac{d^2 f(t)}{dt^2} + 6\frac{df(t)}{dt} + 8 f(t) = 1 \qquad t \geq 0$$

a) Find $f(t)$, $t \geq 0$, without using Laplace transforms

b) Find $f(t)$, $t \geq 0$, using Laplace transforms

1\. Let $f_n = n2^n + 4^n \quad n = 0.1.2.3. \cdots$ Find $F(z)$.

$$F(z) = \sum_{n=0}^{\infty} f_n z^n = \sum_{n=0}^{\infty} (n2^n + 4^n) z^n$$

$$= \sum_{n=0}^{\infty} n(2z)^n + \sum_{n=0}^{\infty} (4z)^n$$

$$= z\left[2\sum_{n=1}^{\infty} n(2z)^{n-1}\right] + \sum_{n=0}^{\infty} (4z)^n$$

$$= z \frac{d}{dz}\left[\sum_{n=0}^{\infty} (2z)^n\right] + \sum_{n=0}^{\infty} (4z)^n$$

$$= \frac{2z}{(1-2z)^2} + \frac{1}{1-4z}$$

$$= \frac{1-2z-4z^2}{(1-4z)(1-2z)^2}$$

2\. $F(z) = \dfrac{2z-1}{-3z^2+4z-1}$ Find f_n. $n = 0.1.2.3. \cdots$

$$F(z) = \frac{2z-1}{-3z^2+4z-1} = \frac{1-2z}{(1-z)(1-3z)} = \frac{1/2}{1-z} + \frac{1/2}{1-3z}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(1+3^n) z^n \quad \Rightarrow \quad f_n = \frac{1}{2}(1+3^n)$$

3\. $G(z) = F(z) + z^2 F(z)$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(1+3^n) z^n + \sum_{n=0}^{\infty} \frac{1}{2}(1+3^n) z^{n+2}$$

$$\Rightarrow \begin{cases} g_0 = f_0 = 1 \\ g_1 = f_1 = 2 \\ g_n = f_n + f_{n-2} = \dfrac{3^n}{2} + \dfrac{3^{n-2}}{2} + 1 \quad \text{for } n \geq 2 \end{cases}$$

or $g_n = \frac{1}{2}\delta_{n-2} + \frac{1}{2}3^{n-2} + \frac{1}{2}\delta_n + \frac{1}{2}3^n$

or $g_n = 5(3^{n-2}) - \frac{2}{3}u_{n-1} - \frac{5}{4}u_n + 1.$

4\. $f(t) = u_0(t) + te^{-t} + e^{2t} \quad t \geq 0$ Find $F^*(s)$

$$F^*(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^-}^{\infty} \left(u_0(t) + te^{-t} + e^{2t}\right)e^{-st}\,dt$$

$$= 1 + \frac{1}{(s+1)^2} + \frac{1}{s-2}$$

5\. $F^*(s) = \dfrac{s(s+1)}{(s+2)(s+3)}$ Find $f(t)$.

$$F^*(s) = \frac{s(s+1)}{(s+2)(s+3)} = 1 - \frac{4s+6}{(s+2)(s+3)} = 1 + \frac{2}{s+2} - \frac{6}{s+3}$$

$$f(t) = u_0(t) + 2e^{-2t}\,\delta(t) - 6e^{-3t}\,\delta(t)$$

6. $10 f_n - 7 f_{n-1} + f_{n-2} = (\frac{1}{3})^n \quad n = 2, 3, 4, 5, \cdots$ and $f_0 = 0$, $f_1 = 3$

(a). Find f_n $n = 0, 1, 2, \cdots$ without using Z-transform

1. Find the homogeneous equation.

The characteristic equation is $10\alpha^2 - 7\alpha + 1 = 0$

$$\alpha = \frac{1}{5}, \frac{1}{2}$$

$$\Rightarrow f_n^{(h)} = C_1 (\tfrac{1}{2})^n + C_2 (\tfrac{1}{5})^n$$ where C_1 and C_2 are constants

2. Find the particular solution:

Guess $f_n^{(p)} = C (\frac{1}{3})^n$ and plug into $10 f_n - 7 f_{n-1} + f_{n-2} = (\frac{1}{3})^n$.

$$C (\tfrac{1}{3})^n - 7 C (\tfrac{1}{3})^{n-1} + C (\tfrac{1}{3})^{n-2} = (\tfrac{1}{3})^n \Rightarrow C = -\tfrac{1}{2}$$

$$\Rightarrow f_n^{(p)} = -\tfrac{1}{2} (\tfrac{1}{3})^n$$

3. $\therefore f_n = f_n^{(h)} + f_n^{(p)} = C_1 (\frac{1}{2})^n + C_2 (\frac{1}{5})^n - \frac{1}{2} (\frac{1}{3})^n$

with initial conditions $f_0 = 0$ and $f_1 = 3$

$$\Rightarrow \begin{cases} f_0 = C_1 + C_2 - \frac{1}{2} = 0 \\ f_1 = C_1 \frac{1}{2} + C_2 \frac{1}{5} - \frac{1}{6} = 3. \end{cases} \quad \therefore C_1 = -\frac{175}{18} \quad C_2 = \frac{92}{9}$$

$\Rightarrow$ The final solution is $f_n = \frac{92}{9} (\frac{1}{2})^n - \frac{175}{18} (\frac{1}{5})^n - \frac{1}{2} (\frac{1}{3})^n$

(b) Find f_n $n = 0, 1, 2, \cdots$ using Z-transform

Let $F(z) = \sum_{n=0}^{\infty} f_n z^n$

$$\Rightarrow \sum_{n=2}^{\infty} (10 f_n - 7 f_{n-1} + f_{n-2}) z^n = \sum_{n=2}^{\infty} (\tfrac{1}{3})^n z^n$$

$$10 (F(z) - f_0 - f_1 z) - 7z (F(z) - f_0) + z^2 F(z) = \frac{z^2}{9} \frac{1}{1 - \frac{z}{3}} \quad \because f_0 = 0 \text{ and } f_1 = 3$$

$$F(z) (10 - 7z + z^2) = 30z + \frac{z^2}{3(3-z)}$$

$$F(z) = \frac{270z - 89z^2}{3(3-z)(5-z)(2-z)} = \frac{92/9}{1 - \frac{z}{2}} - \frac{175/18}{1 - \frac{z}{5}} - \frac{1/2}{1 - \frac{z}{3}}$$

$$\Rightarrow \therefore f_n = \frac{92}{9} (\tfrac{1}{2})^n - \frac{175}{18} (\tfrac{1}{5})^n - \tfrac{1}{2} (\tfrac{1}{3})^n \quad \text{for } n = 0, 1, 2, 3, \cdots$$

7. $$\frac{d^2 f(t)}{dt^2} + 6\frac{d f(t)}{dt} + 8 f(t) = 1 \qquad t \geq 0$$

(a) Find $f(t)$. $t \geq 0$, without using Laplace transform.

1. Find the homogeneous equation

The characteristic equation is $\lambda^2 + 6\lambda + 8 = 0$. $\lambda_1 = -2, -4$.

$\Rightarrow \quad f^{(h)}(t) = C_1 e^{-2t} + C_2 e^{-4t}$ where C_1 and C_2 are constants

2. Find the particular equation :

Guess $\overset{(p)}{f}(t) = C$ and plug into original equation.

$8C = 1 \qquad C = \frac{1}{8}$

$\Rightarrow \quad f^{(p)}(t) = \frac{1}{8}$

3. $\therefore \quad f(t) = f^{(h)}(t) + f^{(p)}(t) = C_1 e^{-2t} + C_2 e^{-4t} + \frac{1}{8}$ for $t \geq 0$

(b) Find $f(t)$ $t \geq 0$. using Laplace transform.

Let $F^*(s) = \int_{0^-}^{\infty} f(t) e^{-st}\, dt$

$$\int_{0^-}^{\infty} \left[\frac{d^2 f(t)}{dt} + 6\frac{d f(t)}{dt} + 8 f(t) \right] e^{-st}\, dt = \int_{0^-}^{\infty} e^{-st}\, dt$$

$$\because \quad \frac{d^n f(t)}{dt^n} \Leftrightarrow S^n F^*(s) - S^{n-1} f(0^-) - S^{n-1} f^{(1)}(0^-) \cdots - f^{(n-1)}(0^-)$$

$$\Rightarrow \quad S^2 F^*(s) - S f(0^-) - f^{(1)}(0^-) + 6SF^*(s) - 6f(0^-) + 8F^*(s) = \frac{1}{S}$$

$$F^*(s)\left[S^2 + 6S + 8\right] = \frac{1}{S} + \left[6f(0^-) + f^{(1)}(0^-)\right] + S f(0^-)$$

$$F^*(s) = \frac{1 + S(6f(0^-) + f^{(1)}(0^-)) + S^2 f(0^-)}{S(S+2)(S+4)}$$

$$= \frac{1/8}{S} + \frac{C_1}{S+2} + \frac{C_2}{S+4}$$

where $C_1 = F_1(f(0^-), f^{(1)}(0^-))$ and $C_2 = F_2(f(0^-), f^{(1)}(0^-))$ are all constants

$$\Rightarrow \quad \therefore \quad f(t) = C_1 e^{-2t} + C_2 e^{-4t} + \frac{1}{8}$$