CSC5420
Computer System Performance Evaluation

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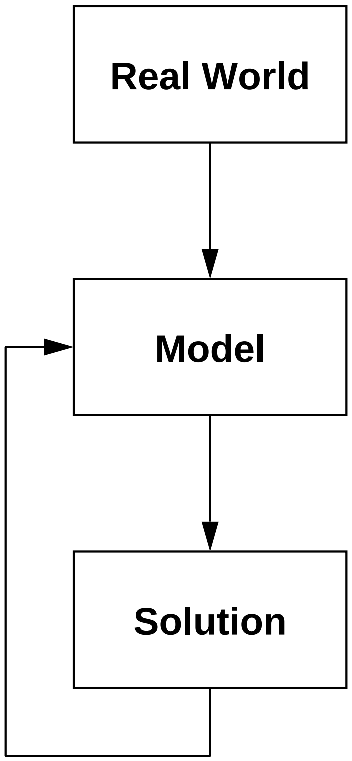
Introduction

- Material from lectures (ref. books on web page)
- Grading
  - homework for students' benefit (will include use of software-tools on web page)
  - 10% homework
  - 40% projects
  - 50% final exam

Course Material

- Review of Probability, R.V., Transforms
- Intro. to Stoch. process (m.c.'s)
- Baby queuery theory m/m/1...
- Intermediate queuery theory m/g/1...
- Markovian model is a special structure
  - Appr. Tech.
  - Stoch. Couple
  - Matrix geometric structure
- Sample Path Analysis
- Transient Analysis
- Reversibility
- Queuery Networks - product form
- Simulation
- Measurements



- **Project: List to choose from, FCFS**
- **MS Comp: Final**

Combinatorics

- **Permutations**
  - ***k*-permutation of a set of *n* elements**

$$\Rightarrow \quad n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

- **Combinations**
  - ***k*-combination of a set of *n* elements**
    - ***k*-permutation / *k!***
      ***k!* is the number of possible ways to permute that combination**

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Combinatorics (Cont...)

- **Binomial Coefficients:**

$$\Rightarrow \binom{n}{k} = \binom{n}{n-k}$$

- **Binomial Expansion:**

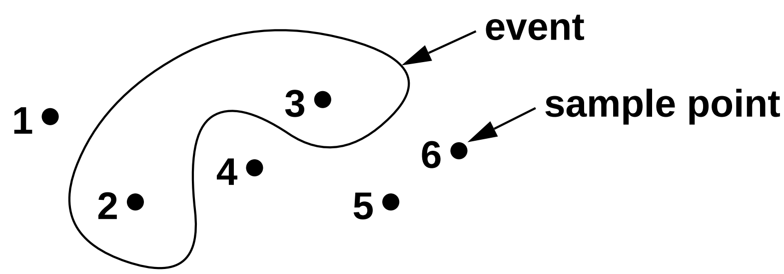
$$\Rightarrow (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Probability

- Sample Space *(S)*, collection of objects, where each object is a sample point.
- A family of events, $\Sigma = \{A, B, C, \ldots\}$ where an event is a set of sample points.
- A probability measure *P* is an assignment (mapping) of events defined on *S* into real numbers (which has properties or axioms).
  - *P[A]* = Probability of event *A*

Probability

- *S* = sample space = set whose elements are *elementary events* (possible outcome of an experiment)
  - The *elementary events* are points in a sample space (1 or more dimensions), and they are *mutually exclusive*
  - Ex: flipping a coin, a elementary events (sample points) H, T
  - *Event:* subset of sample points
    - Ex: toss dice



Probability (Cont...)

- **Aximoms of Probability:**
  - **A *probability distribution* *Pr{}* on a sample space *S* is a mapping from events of *S* to real numbers s.t. the following *proability axioms* hold:**

    1) $Pr\{A\} \geq 0$ for any event *A* (where $Pr\{A\} \equiv$ probability of event *A*)

    2) $Pr\{S\} = 1$

    3) $Pr\{A \cup B\} = Pr\{A\} + Pr\{B\}$ for events *A* and *B* that are *mutually exclusive*

$$\Rightarrow \quad Pr\{\bigcup_i A_i\} = \sum_i Pr\{A_i\}$$

Probability (Cont...)

- **Things that follow:**

  a) $A \subseteq B \;\Rightarrow\; Pr\{A\} \leq Pr\{B\}$

  b) $Pr\{\emptyset\} \;=\; 0$

  c) $\bar{A} \;\equiv\; S - A \;\Rightarrow\; Pr\{\bar{A}\} = 1 - Pr\{A\}$

  d) for any A, B

$$
\begin{aligned}
\Rightarrow \quad Pr\{A \bigcup B\} &= Pr\{A\} + Pr\{B\} - Pr\{A \bigcap B\} \\
&\leq Pr\{A\} + Pr\{B\}
\end{aligned}
$$

Discrete Probability Distribution

- **Probability distribution is discrete if it is defined over a finite or countably infinite sample space**
- $Pr\{A\} = \sum_{x \in A} Pr\{x\}$ **if *x*'s are *mutually exclusive* events in *A***
- **If S is finite and event elementary event in S has probability**

$$Pr\{x\} = \frac{1}{|S|}$$

then we have the *uniform distribution* on *S* (or we pick an element of *S* at random)

Discrete Probability Distribution (Cont...)

- **Ex: flipping a fair coin, *Pr{H}* = *Pr{T}* = *0.5***
  flip coin *n* times
  ***A* = {exactly *k* heads and exactly *n-k* tails}**

$$\Rightarrow \quad A \subseteq S \;\Rightarrow\; |A| = \begin{pmatrix} n \\ k \end{pmatrix} = Pr\{A\} = \begin{pmatrix} n \\ k \end{pmatrix} \cdot \left(\frac{1}{2}\right)^n$$

$$\text{since each outcome } (s \in A) = \left(\frac{1}{2}\right)^n$$

Continuous Uniform Probability Distribution

- Defined over closed interval *[a,b]* of reals where *a < b*
    - (all subsets here, not events)
  - want each point in *[a,b]* to be equally likely
  - but, infinite number of points, if give each one finite probability, will not be able to satisfy axioms 2 and 3
  - associate probability with ***some*** of the subsets
- for any closed interval [c,d], a $\leq$ c $\leq$ d $\leq$ b,
  continuous uniform probability distribution:

$$Pr\{[c,d]\} = \frac{d-c}{b-a}$$

$$(Pr\{[c,d]\} = Pr\{(c,d)\}, \text{ since } Pr\{[x,x]\} = Pr\{x\} = 0)$$

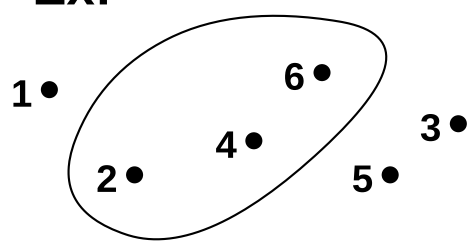
Conditional Probabilities and Independence

- **Def'n:** $$Pr\{A|B\} = \frac{Pr\{A \bigcap B\}}{Pr\{B\}} \text{ whenever } Pr\{B\} \neq 0$$

normalizing (sum to 1)

constrained sample space, so we scale up

- **Ex:**



$$P[x \leq 4|B] = \frac{2}{3} = \frac{\frac{1}{6} + \frac{1}{6}}{\frac{1}{2}}$$

Conditional Probabilities and Independence (Cont...)

- A and B are ***statistically independent*** if and only if:

$$Pr\{A \bigcap B\} = Pr\{A\} \cdot Pr\{B\}$$

- If A_1, A_2, ..., A_n are statistically independent

$$\Rightarrow \quad P\left[A_1 \bigcap A_2 \bigcap \ldots \bigcap A_n\right] = \prod_{i=1}^{n} P\left[A_i\right]$$

- Also, if A and B are statistically independent, then

$$P\left[A|B\right] = \frac{P\left[AB\right]}{P\left[B\right]} = P\left[A\right]$$

Theorem of Total Probability

$$P[B] = \sum_{i=1}^{n} P[B|A_i]\, P[A_i]$$

more useful form

where $\{A_i\}$ is a set of mutually exclusive exhaustive events

$$P[B] = \sum_{i=1}^{n} P[A_i B]$$

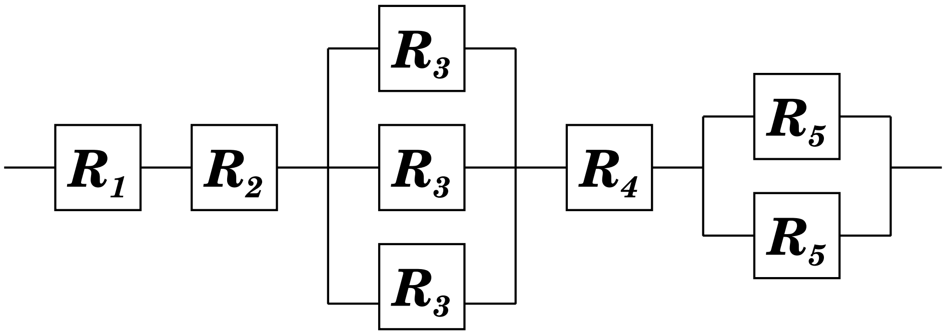
if occurs, occurs with exactly 1 mutually exclusive exhaustive event (A_i)

using conditional probability:

$$P[A_i B] = P[A_i|B]\, P[B] = P[B|A_i]\, P[A_i]$$

Theorem of Total Probability (Cont...)

- **Ex: reliability**



$$R_{sys} = R_1 \cdot R_2 \cdot \left(1 - (1 - R_3)^3\right) \cdot R_4 \cdot \left(1 - (1 - R_5)^2\right)$$

where $\{R_i\}$ is the reliability of component i

- **Importance of Theorem of Total Probability is to break a complex problem into many *simpler problems***

Bayes' Theorem

- **Look at problem from another perspective**
- **Assume we know event *B* has occurred, but we want to find which mutually exclusive event has occurred**

$$P[A_i|B] = \frac{P[B|A_i]\,P[A_i]}{\sum_{j=1}^{n} P[B|A_j]\,P[A_j]}$$

$$P[A \cdot B] = P[B] \cdot P[A|B] = P[A] \cdot P[B|A]$$

$$P[A|B] = \frac{P[A] \cdot P[B|A]}{P[B]}$$

$$P[B] = P[B \cdot A] + P\left[B \cdot \bar{A}\right] = P[A] \cdot P[B|A] + P\left[\bar{A}\right] \cdot P\left[B|\bar{A}\right]$$

$$\Rightarrow P[A|B] = \frac{P[A] \cdot P[B|A]}{P[A] \cdot P[B|A] + P\left[\bar{A}\right] \cdot P\left[B|\bar{A}\right]}$$

more general form above

Bayes' Theorem (Cont...)

- **More general forms:**
  A_i, $1 \leq i \leq n$, are mutually exclusive, exhaustive events

Theorem of total probability $\Rightarrow P[B] = \sum_{i=1}^{n} P[B|A_i] \cdot P[A_i]$

Bayes' Theorem $\Rightarrow P[A_i|B] = \frac{P[B|A_i] \cdot P[A_i]}{\sum_{j=1}^{n} P[B|A_j] \cdot P[A_j]}$

Example

- Ex: gambling, $D_H \Leftarrow$ honest dealer, $D_C \Leftarrow$ cheating dealer
  $L \Leftarrow$ you lose
  play honest dealer $\Rightarrow$ lose with prob = *1/2*
  play cheating dealer $\Rightarrow$ lose with prob = p
  (of p > *1/2* against you, of p < *1/2* for you)

$$
\begin{aligned}
P[D_C|L] &= \frac{P[L|D_C] \cdot P[D_C]}{P[L|D_C] \cdot P[D_C] + P[L|D_H] \cdot P[D_H]} \\
&= \frac{p \cdot \frac{1}{2}}{p \cdot \frac{1}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)} = \frac{2p}{2p+1}
\end{aligned}
$$

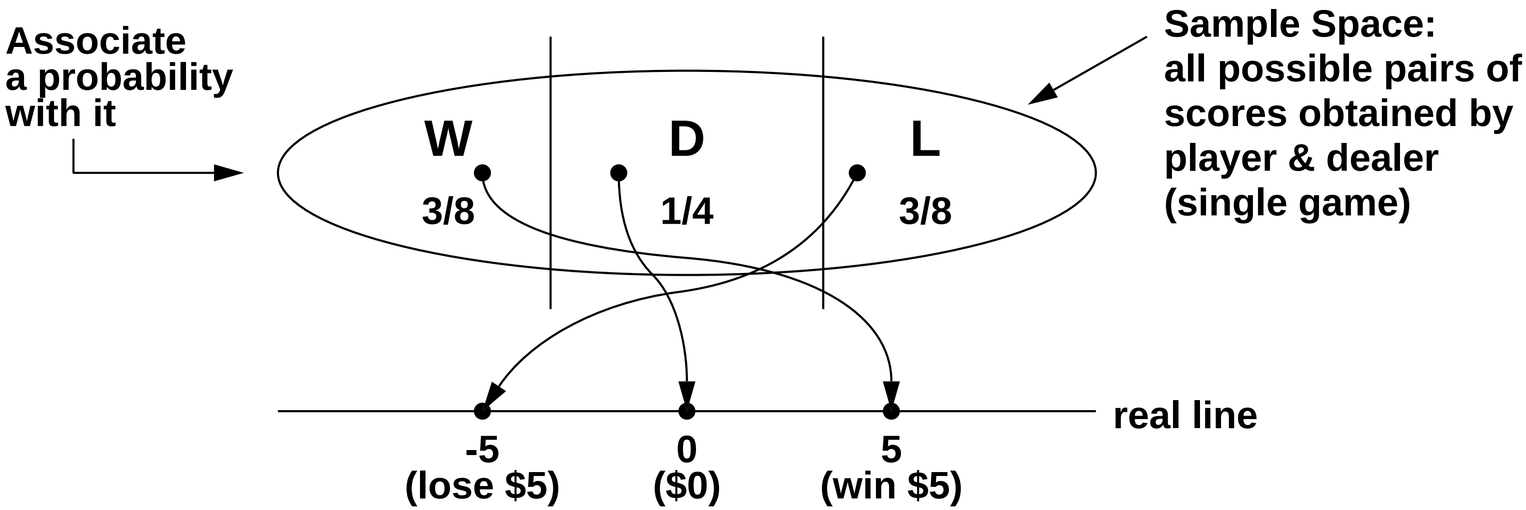
$\Rightarrow$ if p=*1*, prob. that cheating dealer if lost 1 game = *2/3*

Random Variables (R.V.)

- **We have the probability system *(S, Σ, P)***
- **R.V. is a variable whose value depends upon the outcome of the random experiment**
- **The *outcome* of a random experiment is $w \in S$**
  - **We associate a real number *X(w)* with *W***
- **Thus our r.v. *X(W)* is nothing more than a function defined on the sample space *S***
  - **i.e., a function from a finite or countably infinite sample space *S* to real numbers**

Example

- Ex: playing a game of black jack in Las Vegas



- Notation: R.V. X on a sample S

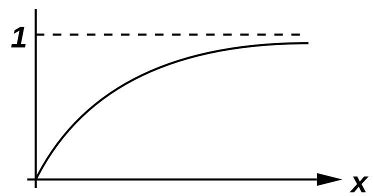
  $X : S \rightarrow R$

  → X(w) - our winnings on a single game of black jack

$$X(w) = \begin{cases} +5 & w \in W \\ 0 & w \in D \\ -5 & w \in L \end{cases}$$

Discrete Random Variables

- **Discrete Random Variables: *(X)***
  - **a function from a finite (or countably infinite) sample sapce S to real numbers**
    - **interested in functions of events**
      - $\Rightarrow$ **each outcome is a combination of events, so we can assign probability to it**
  - ***P[X≤x]:* probability distribution function**



Discrete Random Variable (Cont...)

$$Pr\{X=x\} = \sum_{s\in S; X(s)=x} Pr\{s\}$$

$$f(x) = Pr\{X=x\} \Rightarrow \text{ probability mass function of } X$$

$$\Rightarrow Pr\{X=x\} \geq 0, \ \sum_x Pr\{X=x\} = 1$$

$$f(x,y) = Pr\{X=x \text{ and } Y=y\}$$

is the joint probability mass function of X and Y

$$Pr\{Y=y\} = \sum_x Pr\{X=x \text{ and } Y=y\}$$

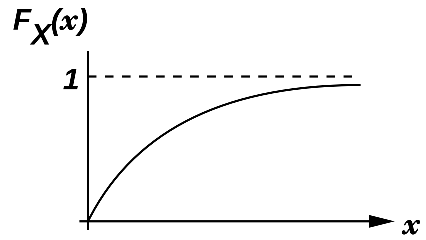
$$Pr\{X=x\} = \sum_y Pr\{X=x \text{ and } Y=y\}$$

$$Pr\{X=x|Y=y\} = \frac{Pr\{X=x \text{ and } Y=y\}}{Pr\{Y=y\}}$$

X and Y are independent if $\forall x, y$:

$$Pr\{X=x \text{ and } Y=y\} = Pr\{X=x\} \cdot Pr\{Y=y\}$$

Expectation, Variance, and Standard Deviation



$$E[X] = \sum_x x \cdot Pr\{X = x\}$$

$$E[X+Y] = E[X] + E[Y]$$

$$E[g(X)] = \sum_x g(x) \cdot Pr\{X = x\}$$

if X and Y are independent, then $E[XY] = E[X]E[Y]$

if X takes on rational numbers $N = \{0, 1, 2, \ldots\}$

$$\Rightarrow \; E[X] = \sum_{i=0}^{\infty} i \cdot Pr\{X = i\} = \sum_{i=1}^{\infty} Pr\{X \geq i\}$$

$$\sigma_X^2 = Var[X] = E\left[(X - E[X])^2\right] = E\left[X^2\right] - (E[X])^2$$

standard deviation $= \sqrt{Var[X]} = \sigma_X$

if X and Y are independent $\Rightarrow Var[X+Y] = Var[X] + Var[Y]$

Probability Distribution Function (PDF) or Cumulative Distribution Function

$$[X \le x] \equiv \{w : X(w) \le x\}$$

PDF is defined as $F_X(x) = P[X \le x]$

Properties:

1) $F_X(x) \ge 0$

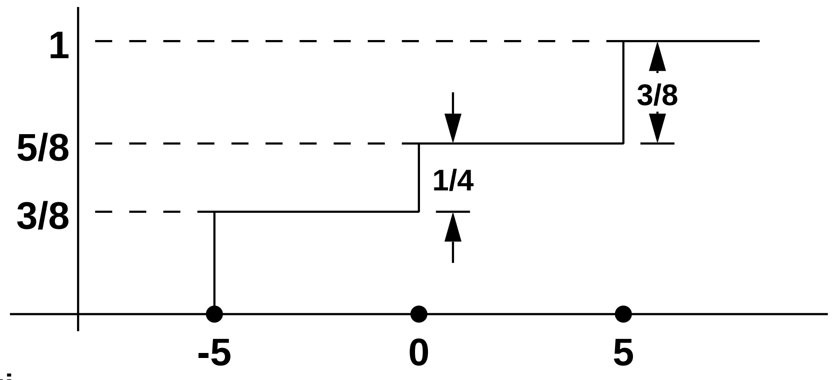
2) $F_X(\infty) = 1$

3) $F_X(-\infty) = 0$

4) $F_X(b) - F_X(a) = P[a < X \le b]$ for $a < b$

5) $F_X(b) \ge F_X(a)$ for $a \le b$

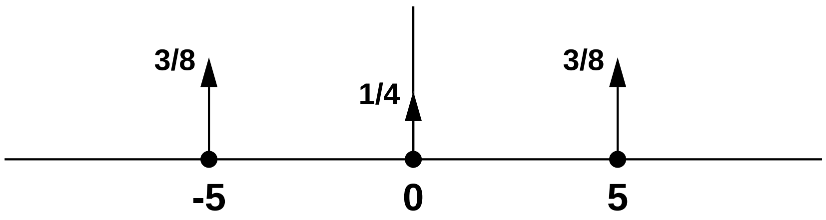
Ex: PDF for the Las Vegas Game



Probability Density Function (pdf) or Probability Mass Function (pmf)

$$f_X = \frac{dF_X(x)}{dx}$$

- **Ex: pdf for the blackjack game**



Different ways to view pdf:

1) $F_X(x) = \int_{-\infty}^{x} f_X(y)dy$

2) $f_X(x) \geq 0$

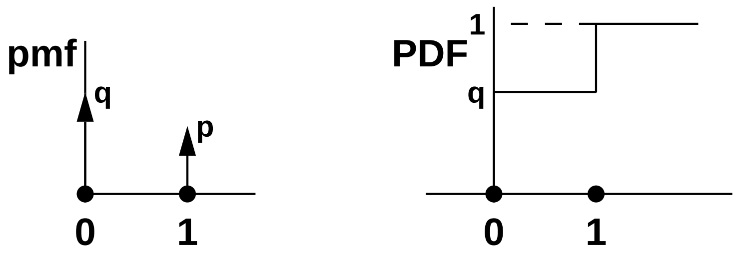
3) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

4) $P\left[a < X \leq b\right] = \int_{a}^{b} f_X(x)dx$

Special Discrete Distribution

- **The Bernoulli pmf**

$$P_X(0) = P\left[X = 0\right] = q$$
$$P_X(1) = P\left[X = 1\right] = p = (1 - q)$$



Geometric Distribution

- **Bernoulli trial:**
  - **experiment with only 2 possible outcomes**
    - **success with probability *p***
    - **failure with probability *q = 1-p***
  - **Bernoulli trials, a sequence of *independent* trials each with probability *p***

$\Rightarrow$ **r.v. *X* = number of trials needed to obtain success**

X $\in$ ***{ 1, 2, ... }***

$$\text{for } k \geq 1, Pr\{X = k\} = q^{k-1}p = (1-p)^{k-1} \cdot p$$

$\Rightarrow$ ***geometric distribution***

$$\text{assume } p \leq 1, \Rightarrow E[X] = \frac{1}{p} = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$$

$\Rightarrow$ **on average, *1/p* trials before obtain success**

$$\sigma_X^2 = \frac{1-p}{p^2}$$

Binomial Distribution

⇒ **r.v. *X* = number of successes in *n* trials, *X* $\in$ *{ 0, 1, ... }***

for $k = 0, 1, \ldots, n$, $Pr\{X = k\} = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$

⇒ ***binomial distribution***

$E[X] = np, \ \sigma_X^2 = np(1-p)$

Multiple R.V.

- Can, of course, define many r.v. on same sample space
- Let ***X*** & ***Y*** be 2 r.v. on some probability system ***(S, Σ, P)***
- Natural extension of PDF:

$$F_{XY}(x, y) \equiv P\left[X \leq x, Y \leq y\right]$$

$\Rightarrow$ ***joint*** **PDF**

- ***Joint*** **probability density function:**

$$f_{XY}(x, y) \equiv \frac{d^2 F_{XY}(x, y)}{dxdy}$$

- ***Marginal*** **density function (for one of the variables):**

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x, y) dy$$

(given by integrating over all possible values of the 2nd variable)

Multiple R.V. (Cont...)

- Notion of ***independence*** between r.v.'s
  - ***X*** **&** ***Y*** **are independent iff: (same for more than 2 variables)**

$$f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$$

- **Can also define one random variable in terms of another, i.e.,**

$$y = g(x)$$

$$\Rightarrow F_Y = P[Y \leq y] = P[\{w : g(X(w)) \leq y\}]$$

(could be complex computation)

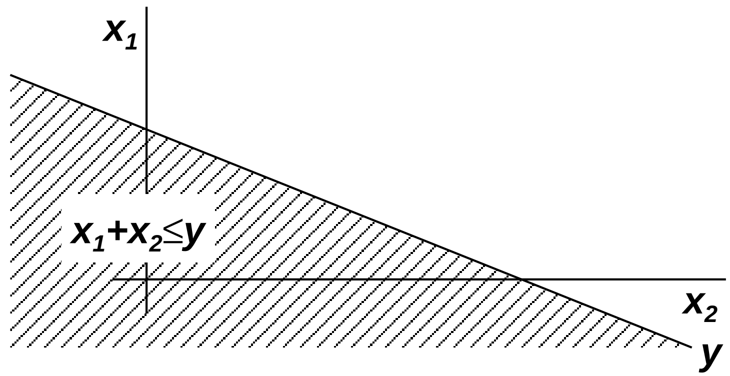
- **Of course, can be a function of many r.v.**

Example

- **Ex: Let $Y = X_1+X_2$ (i.e., sum of 2 r.v.) where X_1 and X_2 are *independent***

$$\Rightarrow F_Y(y) = P\left[Y \leq y\right] = P\left[X_1 + X_2 \leq y\right]$$

$$F_Y(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y-x_2} f_{X_1X_2}(x_1,x_2)dx_1dx_2$$



Example (Cont...)

$\Rightarrow$ **due to independence:**

$$F_Y(y) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{y-x_2} f_{X_1}(x_1) dx_1 \right] f_{X_2}(x_2) dx_2$$
$$= \int_{-\infty}^{\infty} F_{X_1}(y - x_2) f_{X_2}(x_2) dx_2$$

$$\Rightarrow f_Y(y) = \underbrace{\int_{-\infty}^{\infty} f_{X_1}(y - x_2) f_{X_2}(x_2) dx_2}$$

convolution **of density functions of** X_1 **and** X_2

$$\Rightarrow f_Y(y) = f_{X_1}(y) \otimes f_{X_2}(y)$$

(same for any *n* sum of independent r.v.)

Expectation

- **The *expectation* of a real r.v. *X(w)* is denoted by *E[X]***

$$\Rightarrow \text{also denoted by } \bar{X}$$

$$E[X] \equiv \bar{X} \equiv \int_{-\infty}^{\infty} x dF_X(x)$$

$$E[X] = \bar{X} = \int_{-\infty}^{\infty} x f_X(x) dx$$

$$E[X] = \int_{0}^{\infty} [1 - F_X(x)]\, dx - \int_{-\infty}^{0} F_X(x) dx \longrightarrow \textbf{Stieltjes Integral}$$

- **For *X*, a nonnegative r.v.**

$$E[X] = \int_{0}^{\infty} [1 - F_X(x)]\, dx \qquad x \geq 0$$

Fundementation Theorem of Expectation

- Let *y=g(x)*

$$E_Y[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy$$

could be complex to compute $f_Y(y)$

- ***Fundementation Theorem of Expectation:***

$$E_Y[Y] = E_X[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- **Expectation of sum of 2 r.v.**

$$\begin{aligned} E[X+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y) f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)dxdy \\ &= \int_{-\infty}^{\infty} x f_X(x)dx + \int_{-\infty}^{\infty} y f_Y(y)dy \\ &= E[X] + E[Y] \end{aligned}$$

(generalize to any number of variables)

  - True whether or not *X* & *Y* are independent

Product of R.V.

$$E\left[XY\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy$$

- **If *X* & *Y* are *independent*, then**

$$E\left[XY\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_X(x) y f_Y(y)dxdy = E\left[X\right]\cdot E\left[Y\right]$$

$$\Rightarrow \quad E\left[g(X)h(Y)\right] = E\left[g(X)\right]\cdot E\left[h(Y)\right]$$

- **Interested in power of r.v.'s**

$$\Rightarrow \; E\left[X^n\right] \Rightarrow \; \mathrm{n}^{th} \text{ moment of X}$$

$$\Rightarrow \; E\left[X^n\right] \equiv \bar{X^n} \equiv \int_{-\infty}^{\infty} x^n f_X(x)dx$$

follows from the fundemental theorem of expectation

$$\Rightarrow \; n^{th} central moment is:$$

$$\left(X - \bar{X}\right)^n \equiv \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x)dx$$

$$\Rightarrow \; \left(X - \bar{X}\right)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^k (-\bar{X})^{n-k}$$

binomial theorem

Product of R.V. (Cont...)

take expectation of both sides

$$\Rightarrow \overline{\left(X-\bar{X}\right)^n} = \sum_{k=0}^{n} \binom{n}{k} \overline{x^k}(-\bar{X})^{n-k}$$

sums of expectation, expectation of sums, expectation of constant

$$\Rightarrow 0^{th} \text{ moment } = 1; \text{ also } 0^{th} \text{ central moment } = 1$$

$$\Rightarrow 1^{st} \text{ central moment } \Rightarrow \overline{\left(X-\bar{X}\right)} = \bar{X}-\bar{X} = 0$$

$$\Rightarrow 2^{nd} \text{ central moment } \Rightarrow \text{ variance}$$

$$\sigma_X^2 \equiv \overline{\left(X-\bar{X}\right)^2} \equiv \overline{X^2} - \left(\bar{X}\right)^2$$

$$\sigma_X \Rightarrow \text{ standard deviation } = \sqrt{\sigma_X^2}$$

$$\Rightarrow C_X \equiv \frac{\sigma_X}{\bar{X}}$$

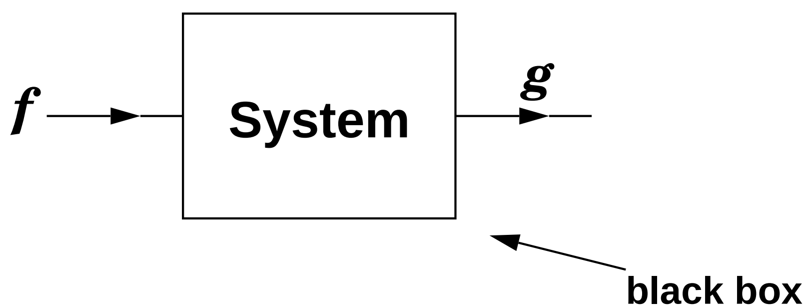
coefficient of variation

Transforms

- Transforms, characteristic function, generating function...
- Laplace, z, Fourior, ...
- When introduce into solution method, simplify calculations
- Appear naturally, why?

Linear Time-invariant Systems

systems $\equiv$ transformations, mapping, input-output relationship between 2 functions



assume $\Rightarrow$ $f = f(t)$, $f(t) \rightarrow g(t)$

Linear and Time-invariant

- *Linear* if when $f_1(t) \rightarrow g_1(t)$ and $f_2(t) \rightarrow g_2(t)$

  then also $af_1(t)+bf_2(t) \rightarrow ag_1(t)+bg_2(t)$

- *Time-invariant* if when $f(t) \rightarrow g(t)$

  then also $f(t+\tau) \rightarrow g(t+\tau)$

- If both holds, we have a *linear time-invariant system*

we focus on these

Transforms

- **Decompose function of time into sums (integrals) of complex exponentials**
  - **complex exponentials form building blocks of transforms**

- ***Question:* which functions of time can pass through linear time-invariant systems without change?**
  - **i.e., $f(t) \to g(t) = Hf(t)$, where H is some scalar multiple**
  - **if can find these, then can find *eigenfunctions* or *characteristic functions*, or *invariants* of our system**

→ $f_e(t) \;=\; e^{st}$ **where s is a complex variable**

→ **form the set of eigenfunctions for *all* linear time-invariant systems**

Characteristic Functions

- **Derivation:**

linearity

$$f_e(t) = e^{st} \rightarrow g_e(t)$$

$$e^{s\tau} f_e(t) = e^{s(t+\tau)} \rightarrow e^{s\tau} g_e(t)$$

where τ and hence $e^{s\tau}$ are constant

time-invariant

$$f_e(t+\tau) = e^{s(t+\tau)} \rightarrow g_e(t+\tau)$$

$$\Rightarrow e^{s\tau} g_e(t) = g_e(t+\tau)$$

unique solution

$$g_e(t) = H e^{st}$$

$$e^{st} \rightarrow H(s)\, e^{st}$$

independent of *t* but can be function of *s*

Characteristic Functions (Cont...)

- **Overall output found by summing (integrating) these individual components of the output**
  - **decompose input into sums of exponentials, computing response to each as above, and then reconstituting the output from sums of exponentials is referred to as *transform* method**

Transforms

- **Focus on discrete time first**

$$f(t) \;=\; f(t = nT) \qquad \text{where } n = \ldots, -2, -1, 0, 1, 2, \ldots$$

$\hookrightarrow f_n$

$$\Rightarrow \; f_n \to g_n$$

$$af_n^{(1)} + bf_n^{(2)} \;\to\; ag_n^{(1)} + bg_n^{(2)}$$

$$f_{n+m} \to g_{n+m} \qquad m \text{ is an integer constant}$$

$\Rightarrow$ ***eigenfunctions:***

$$f_n^{(e)} = e^{st} = e^{snT}$$

$$\text{Let } z \equiv e^{-sT} \;\Rightarrow\; f_n^{(e)} = z^{-n}$$

$\hookrightarrow$ **also a complex variable**

Transforms (Cont...)

$$\Rightarrow \; z^{-n} \;\rightarrow\; H(z)\, z^{-n} \qquad H(z) \text{ is independent of } n \qquad\qquad (\text{t1})$$

$$\Rightarrow \; \left\{ z^{-n} \right\} \text{ form a set of eigenfunctions}$$

$\Rightarrow$ ***H*** **expresses how much we get out of unit input** $\Rightarrow$ **system (or transfer) function**

- ***Kronecker delta function*** **or** ***unit function*****:**

$$u_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

- ***Unit response*** **(when apply** $\boldsymbol{u_n}$ **to system),** $\boldsymbol{h_n}$

$$u_n \;\rightarrow\; h_n$$

$$\Rightarrow \; u_{n+m} \;\rightarrow\; h_{n+m}$$

Transforms (Cont...)

linearity $\Rightarrow \quad z^m u_{n+m} \;\rightarrow\; z^m h_{n+m}$

multiple by unity on both sides $\Rightarrow \quad \left(z^{-n} z^n\right) z^m u_{n+m} \;\rightarrow\; \left(z^{-n} z^n\right) z^m h_{n+m} \qquad \text{(t2)}$

- **Consider set of inputs $\{f_n^{(i)}\}$**

$$\Rightarrow \quad f_n^{(i)} \;\rightarrow\; g_n^{(i)}$$

linearity $\Rightarrow \quad \sum_i f_n^{(i)} \;\rightarrow\; \sum_i g_n^{(i)} \qquad \Rightarrow \text{ apply to Eq. (t2)}$

$$\Rightarrow \quad z^{-n} \underbrace{\sum_m z^{n+m} u_{n+m}} \;\rightarrow\; z^{-n} = \sum_m z^{n+m} h_{n+m}$$

sum over all integer value of *m*

only 1 non-zero term, when m=-n, and it equals 1

Transforms (Cont...)

plus change of variables

$$\Rightarrow z^{-n} \rightarrow z^n \sum_k z^k h_k \qquad \text{(go back to Eq. (t1))}$$

$$\Rightarrow H(z) = \sum_k h_k z^k$$

- **related system function *H(z)* to unit response**
- **both *H(z)* and unit response describe how the system operates, so they are related**
- **itself a transform, a *z-Transform***
  - **so transforms arise naturally**

z-Transform

- Let f_n be a function which takes on nonzero values
  - only for non-negative integers, n=0, 1, 2, ... (f_n=0 for n < 0)
- Compress sequence into a single function such that can expand later
- Place a tag on each term
  - i.e., tag each f_n with z^n (n unique $\Rightarrow$ each tage is unique)

$\Rightarrow$ Define z-transform (or generating function, or geometric transform)

$$F(z) \equiv \sum_{n=0}^{\infty} f_n z^n$$

$\Rightarrow$ The z-transform will exist as long as terms don't grow any faster than geometrically, i.e., as long as a exists, s.t.

$$\lim_{n \to \infty} \frac{|f_n|}{a^n} = 0$$

z-Transform (Cont...)

$\Rightarrow$ **Given a sequence f_n, its z-transform is unique**

If sum over all f_n is finite, then $F(z)$ is analytic on $|z| \leq 1$.

Then:

$$\Rightarrow \quad F(1) = \sum_{n=0}^{\infty} f_n$$

- **has a unique derivative at that point $\Rightarrow$ function is analytic at that point**

Notation:

$$f_n \;\Leftrightarrow\; F(z)$$

Examples of z-Transforms

- **Ex 1: Recall the unit function**

$$u_n = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$\Rightarrow \text{ Exactly 1 term in the infinite summation is non-zero}$$

$$\Rightarrow u_n \Leftrightarrow 1$$

- **Ex 2: Shift the unit function to the right**

$$u_{n-k} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

$$\Rightarrow u_{n-k} \Leftrightarrow z^k$$

Examples of z-Transforms (Cont...)

- **Ex 3: unit step function**

$$\delta_n = 1 \quad \text{for } n = 0, 1, 2, \ldots$$

$$\delta_n \;\Leftrightarrow\; \sum_{n=0}^{\infty} 1 \cdot z^n \;=\; \frac{1}{1-z} \;\Rightarrow\; |z| < 1 \text{ for transform to exist}$$

- **Ex 4: geometric series**

$$f_n = A\alpha^n \quad n = 0, 1, 2, \ldots$$

$$\Rightarrow\; F(z) = \sum_{n=0}^{\infty} A\alpha^n \cdot z^n = A \sum_{n=0}^{\infty} (\alpha z)^n = \frac{A}{1-\alpha z}$$

$$A\alpha^n \;\Leftrightarrow\; \frac{A}{1-\alpha z} \quad (\text{where } |z| < \frac{1}{\alpha})$$

region of analycity

Properties of z-Transforms

- **Convolution property**
  - **We have 2 function, f_n and g_n with $f_n \Leftrightarrow F(z)$ and $g_n \Leftrightarrow G(z)$**

$$f_n \otimes g_n \equiv \sum_{k=0}^{n} f_{n-k} g_k$$

$$\Rightarrow f_n \otimes g_n \Leftrightarrow \sum_{n=0}^{\infty} (f_n \otimes g_n) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} f_{n-k} g_k z^{n-k} z^k$$

$$\text{since} \sum_{n=0}^{\infty} \sum_{k=0}^{n} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty}$$

$$\Rightarrow f_n \otimes g_n \Leftrightarrow \sum_{k=0}^{\infty} g_k z^k \cdot \sum_{n-k}^{\infty} f_{n-k} z^{n-k} = \left(\sum_{k=0}^{\infty} g_k z^k \right) \left(\sum_{m=0}^{\infty} f_m z^m \right)$$

$$= G(z) \cdot F(z)$$

$$\boxed{f_n \otimes g_n \Leftrightarrow G(z) \cdot F(z)}$$

Other Properties of z-Transforms

$$\Rightarrow \quad af_n + bg_n \;\Leftrightarrow\; aF(z) + bG(z)$$ → **linearity**

$$\left[\begin{array}{l} \Rightarrow \quad a^n f_n \;\Leftrightarrow\; F(az) \\ \Rightarrow \quad f_n/k \;(n = 0, k, 2k, \ldots) \;\Leftrightarrow\; F(z^k) \end{array}\right.$$

scale change in the transform and time domains

$$\Rightarrow \qquad f_{n+1} \;\Leftrightarrow\; \frac{1}{z}\left[F(z) - f_0\right]$$

$$f_{n+k} \quad (k > 0) \;\Leftrightarrow\; \frac{F(z)}{z^k} - \sum_{i=1}^{k} z^{i-k-1} f_{i-1}$$

$$\Rightarrow \qquad f_{n-1} \;\Leftrightarrow\; zF(z)$$

advance or delay by unit of time results in divide or multiply in transform domain

Inverse Transforms

- Given $F(z)$, find sequence f_n
  - Power series method, e.g.,

$$f_n = \left. \frac{1}{n!} \frac{d^n F(z)}{dz^n} \right|_{z=0}$$

    - not as useful if want many terms
      $\Rightarrow$ ratio of numerators and denominators
  - Inspection method
    - express $F(z)$ in terms which have recognizable transform pairs
    - *partial-fraction expansion*

Inverse Transforms (Cont...)

- **e.g., each term is either**
  1) a simple pole $\longrightarrow A\alpha^n \Leftrightarrow \dfrac{A}{1-\alpha z}$
  2) a multiple pole

$$\frac{1}{m!}(n+m)(n+m-1)\cdots(n+1)\alpha^n \Leftrightarrow \frac{1}{(1-\alpha z)^{m+1}}$$

in addition, sum of transforms is transform of sums

$$\Rightarrow \quad F(z) = \frac{N(z)}{D(z)}$$

where $N(z) + D(z)$ are polynomials in z and degree of $N(z) <$ degree of $D(z)$

also $\Rightarrow$ $D(z)$ is in factored form

$$D(z) = \prod_{i=1}^{k} (1-\alpha_i z)^{m_i}$$

i^{th} root at $z = \dfrac{1}{\alpha_i}$ and it's of multiplicity m_i

factoring could be the hard part

Inverse Transforms (Cont...)

$$\begin{aligned}\Rightarrow\ F(z) &= \frac{A_{11}}{(1-\alpha_1 z)^{m_1}} + \frac{A_{12}}{(1-\alpha_1 z)^{m_1-1}} + \cdots + \frac{A_{1m_1}}{(1-\alpha_1 z)} \\ &+ \frac{A_{21}}{(1-\alpha_2 z)^{m_2}} + \frac{A_{22}}{(1-\alpha_2 z)^{m_2-1}} + \cdots + \frac{A_{2m_2}}{(1-\alpha_2 z)} \\ &+ \ldots \\ &+ \frac{A_{k1}}{(1-\alpha_k z)^{m_k}} + \frac{A_{k2}}{(1-\alpha_k z)^{m_k-1}} + \cdots + \frac{A_{km_k}}{(1-\alpha_k z)}\end{aligned}$$

use known transform

$$\sum_{k=0}^{n} f_k \Leftrightarrow \frac{F(z)}{1-z}x^2 \qquad (n = 0, 1, 2, \ldots)$$

$$\Rightarrow\ A_{ij} = \frac{1}{(j-1)!}\left(\frac{-1}{\alpha_i}\right)^{j-1} \frac{d^{j-1}}{dz^{j-1}}\left[(1-\alpha_i z)^{m_i}\frac{N(z)}{D(z)}\right]\Bigg|_{z=\frac{1}{\alpha_i}}$$

Example

$$F(z) = \frac{4z^2(1-8z)}{(1-4z)(1-2z)^2}$$

$\Rightarrow$ **need numerator power < denominator power**

$$\text{Let } G(z) = \frac{4(1-8z)}{(1-4z)(1-2z)^2}$$

$\Rightarrow$ **2 poles:** **1)** $z = \frac{1}{4}$

2) $z = \frac{1}{2}$ $\Leftarrow$ **two poles here**

$$\Rightarrow k = 2, \quad \alpha_1 = 4, m_1 = 1, \quad \alpha_2 = 2, m_2 = 2$$

Example (Cont...)

$$\Rightarrow G(z) \equiv \frac{4\ (1-8z)}{(1-4z)(1-2z)^2}$$

ungroup previous equation

$$= \frac{A_{11}}{(1-4z)} + \frac{A_{21}}{(1-2z)^2} + \frac{A_{22}}{(1-2z)}$$

$$A_{11} = (1-4z)G(z)\Big|_{z=\frac{1}{4}} = \frac{4\ (1-\frac{8}{4})}{(1-\frac{2}{4})^2} = -16$$

$$A_{21} = (1-2z)^2 G(z)\Big|_{z=\frac{1}{2}} = \frac{4\ (1-\frac{8}{2})}{(1-\frac{4}{2})} = 12$$

$$A_{22} = -\frac{1}{2}\ \frac{d}{dz}\left[(1-2z)^2 G(z)\right]\Big|_{z=\frac{1}{2}} = -\frac{1}{2}\ \frac{d}{dz}\frac{4\ (1-8z)}{(1-4z)}\Big|_{z=\frac{1}{2}}$$

$$= -\frac{4}{2}\ \frac{(1-4z)(-8)-(1-8z)(-4)}{(1-4z)^2}\Big|_{z=\frac{1}{2}} = 8$$

$$\Rightarrow G(z) = \frac{-16}{(1-4z)} + \frac{12}{(1-2z)^2} + \frac{8}{(1-2z)}$$

Example (Cont...)

by inspection

$$\Rightarrow \; G(z) \;\Leftrightarrow\; g_n = \begin{cases} 0 & n < 0 \\ -16(4)^n + 12(n+1)(2)^n + 8(2)^n & n = 0, 1, 2, \cdots \end{cases}$$

$\Rightarrow$ **need to account for** z^2

$$\Rightarrow \; k > 0, \; f_{n-k} \;\Leftrightarrow\; z^k F(z)$$

$$\Rightarrow \; f_n = -16(4)^{n-2} + 12(n-1)(2)^{n-2} + 8(2)^{n-2}$$

$$\Rightarrow \; f_n = \begin{cases} 0 & n < 2 \\ (3n-1)2^n - 4^n & n = 2, 3, 4, \cdots \end{cases}$$

Laplace Transform

- Consider function of continuous time *f(t)*, *f(t)* = *0, for t* < *0*
- As before, want to transform from a function of *t* to a function of a complex variable *s*, and also want to be able to "untransform", so want "tag" each value *f(t)*
- use e^{-st} as our tag
- $s = \sigma + j\omega$ where $j = \sqrt{-1}$

$$\Rightarrow \; F^*(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

since assume *f(t)* = *0* for *t<0*

$$F^*(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

$0-$ → any accumulation at origin (e.g., impluse function) will be included

$\Rightarrow$ Exists as long as *f(t)* grown no faster than exponential, i.e., there is some real number σ_a s.t.

$$\lim_{T \to \infty} = \int_{0}^{T} |f(t)|\, e^{-\sigma_a t}\, dt < \infty$$

Laplace Transform (Cont...)

⇒ **Laplace transform for a given $f(t)$ is unique**

⇒ **If integral of $f(t)$ is finite, then $Re(s) \geq 0$ represents region of analyticity for $F^*()$**

$$\Rightarrow \; F^*(0) = \int_0^\infty f(t)\, dt$$

($z = 1$ corresponds to $s = 0$)

⇒ **Use notation:**

$$f(t) \Leftrightarrow F^*(s)$$

⇒ **Inverse by inspection**

Examples of Laplace Transforms

- **Ex 1:**

$$f(t) = \begin{cases} A\ e^{-at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$f(t) \Leftrightarrow F^*(s) = \int_0^\infty A\ e^{-at}\ e^{-st}\ dt = A \int_0^\infty e^{-(s+a)t}\ dt$$

$$= \frac{A}{s+a}$$

$$\Rightarrow A\ e^{-at}\ \delta(t) \Leftrightarrow \frac{A}{s+a}$$

$$\text{where } \delta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

unit step function in continuous time (to get *f(t)* defined above)

- **Ex 2:**

$$\text{if } A = 1,\ a = 0 \Rightarrow \text{have unit step function}$$

$$\Rightarrow \delta(t) \Leftrightarrow \frac{1}{s}$$

Inspection Method

- **Assume $F^*(s)$ is a rational function of s, i.e.,**

$$F^*(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in s

degree of $N(s)$ < degree of $D(s)$

$\Rightarrow$ **factor $D(s)$**

$$D(s) = \prod_{i=1}^{k} (s + a_i)^{m_i}$$

Inspection Method (Cont...)

$$
\left.\begin{aligned}
\Rightarrow \quad F^*(s) \quad &= \quad \frac{B_{11}}{(s+a_1)^{m_1}} + \frac{B_{12}}{(s+a_1)^{m_1-1}} + \cdots + \frac{B_{1m_1}}{(s+a_1)} \\
&+ \quad \frac{B_{21}}{(s+a_2)^{m_2}} + \frac{B_{22}}{(s+a_2)^{m_2-1}} + \cdots + \frac{B_{2m_2}}{(s+a_2)} \\
&+ \quad \ldots \\
&+ \quad \frac{B_{k1}}{(s+a_k)^{m_k}} + \frac{B_{k2}}{(s+a_k)^{m_k-1}} + \cdots + \frac{B_{km_k}}{(s+a_k)}
\end{aligned}\right\} \text{ from table}
$$

$$
\text{where } B_{ij} = \frac{1}{(j-1)!} \frac{d^{j-1}}{ds^{j-1}} \left[(s+a_i)^{m_i} \frac{N(s)}{D(s)} \right] \Bigg|_{s=-a_i}
$$

Example

$$
\begin{aligned}
F^*(s) &= \frac{8\ (s^2+3s+1)}{(s+3)(s+1)^3} \\
&\Rightarrow k=2, \quad a_1 = 3, m_1 = 1, \quad a_2 = 1, m_2 = 3
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow F^*(s) &= \frac{B_{11}}{(s+3)} + \frac{B_{21}}{(s+1)^3} + \frac{B_{22}}{(s+1)^2} + \frac{B_{23}}{(s+1)} \\
B_{11} &= (s+3)F^*(s)\big|_{s=-3} = \frac{8\ (9-9+1)}{(-2)^3} = -1 \\
B_{21} &= (s+1)^3 F^*(s)\Big|_{s=-1} = \frac{8\ (1-3+1)}{(2)} = -4 \\
B_{22} &= \frac{d}{ds}\left[\frac{8\ (s^2+3s+1)}{(s+3)}\right]\Bigg|_{s=-1} \\
&= 8\ \frac{(s+3)(2s+3)-(s^2+3s+1)(1)}{(s+3)^2}\Bigg|_{s=-1}
\end{aligned}
$$

Example (Cont...)

$$= \quad 8 \left[\frac{s^2+6s+8}{(s+3)^2}\right]\Bigg|_{s=-1} = 8 \left[\frac{1-6+8}{(2)^2}\right] = 6$$

already took this derirative

$$B_{23} \quad = \quad \frac{1}{2!}\,\frac{d^2}{ds^2}\left[\frac{8\,(s^2+3s+1)}{(s+3)}\right]\Bigg|_{s=-1} = \frac{8}{2}\,\frac{d}{ds}\left[\frac{(s^2+6s+8)}{(s+3)^2}\right]\Bigg|_{s=-1}$$

$$= \quad 4\,\frac{(s+3)^2(2s+6)-(s^2+6s+8)(2)(s+3)}{(s+3)^4}\Bigg|_{s=-1}$$

$$= \quad 4\,\frac{(2)^2(4)-(1-6+8)(2)(2)}{(2)^4} = 1$$

$$\Rightarrow \quad F^*(s) \quad = \quad \frac{-1}{(s+3)} + \frac{-4}{(s+1)^3} + \frac{6}{(s+1)^2} + \frac{1}{(s+1)}$$

using table

$$\Rightarrow \quad f(t) \quad = \quad -e^{-3t} - 2t^2e^{-t} + 6te^{-t} + e^{-t}$$

$$\text{and } f(t) = 0 \text{ for } t < 0$$

Difference Equations

⇒ Nth **order difference equation:** **(standard method)**

$$a_N\ g_{n-N}\ +\ a_{N-1}\ g_{n-N+1}\ +\ \cdots\ +\ a_0\ g_n\ =\ e_n$$

where a_i are the known constants and

g_i are the unknown functions to be found, and

e_n is a given function of n

plus we are given N bounding equations

⇒ **as usual, solution has homogeneous and particular part:**

$$g_n\ =\ g_n^{(h)}\ +\ g_n^{(p)}$$

⇒ **homogeneous solution must satisfy homogeneous equation:**

$$a_N\ g_{n-N}\ +\ a_{N-1}\ g_{n-N+1}\ +\ \cdots\ +\ a_0\ g_n\ =\ 0$$

⇒ **general form of solution is:**

$$g_n^{(h)} = A\ \alpha^n$$

where A and α are to be determined

⇒ $a_N\ A\ \alpha^{n-N}\ +\ a_{N-1}\ A\ \alpha^{n-N+1}\ +\ \cdots\ +\ a_0\ A\ \alpha^n\ =\ 0$

Difference Equations (Cont...)

$\Rightarrow$ ***Nth*** **order polynomial has** $\boldsymbol{N}$ **solutions:**

$$\Rightarrow \ \alpha_1, \ \alpha_2, \ \cdots \ , \alpha_N$$

(assume for now that α_i are distinct)

$\Rightarrow$ A_i**'s are determined from initial conditions**

$\Rightarrow$ **by cancelling common terms, get characteristic equation:**

$$a_N \ + \ a_{N-1} \ A \ \alpha \ + \ a_{N-2} \ A \ \alpha^2 \ + \ \cdots \ + \ a_0 \ A \ \alpha^N \ = \ 0$$

$\Rightarrow$ **find roots of $\nearrow$, if all α_i are distinct, then**

$$g_n^{(h)} = A_1 \ \alpha_1^n \ + \ A_2 \ \alpha_2^n \ + \ \cdots \ + \ A_N \ \alpha_N^n$$

$\Rightarrow$ **with α_1 as a multiple root of order $\boldsymbol{k}$,**

$$(A_{11}n^{k-1} + A_{12}n^{k-2} + \cdots + A_{1k-1}n + A_{1k})\alpha_1^n$$

$\Rightarrow$ $\boldsymbol{g_n^{(p)}}$ **is determined by approp.-given from the form $\boldsymbol{e_n}$**

Example

$$6g_n - 5g_{n-1} + g_{n-2} = 6\left(\frac{1}{5}\right)^n \quad n = 2,3,4,\cdots \quad (*)$$

$$\text{with } g_0 = 0, g_1 = \frac{6}{5}$$

homogen. sol.

$$\Rightarrow \quad 6\alpha^2 - 5\alpha + 1 = 0 \qquad \text{(char. eq.)}$$

$$\Rightarrow \quad \alpha_1 = \frac{1}{2} \quad \alpha_2 = \frac{1}{3}$$

$$\Rightarrow \quad g_n^{(h)} = A_1\left(\frac{1}{2}\right)^n + A_2\left(\frac{1}{3}\right)^n$$

guess

$$\Rightarrow \quad g_n^{(p)} = B\left(\frac{1}{5}\right)^n \quad \Rightarrow \text{ plug into } (*) \quad \Rightarrow \text{ get } B = 1$$

$$\Rightarrow \quad g_n = g_n^{(h)} + g_n^{(p)} = A_1\left(\frac{1}{2}\right)^n + A_2\left(\frac{1}{3}\right)^n + \left(\frac{1}{5}\right)^n$$

using init. cond.

$$\Rightarrow \quad A_1 = 8, A_2 = -9$$

$$\Rightarrow \quad g_n = \left(\frac{1}{2}\right)^{n-3} - \left(\frac{1}{3}\right)^{n-2} + \left(\frac{1}{5}\right)^n \quad n = 0,1,2,\cdots$$

Use z-Tranform

$$a_N\ g_{n-N}\ +\ a_{N-1}\ g_{n-N+1}\ +\ \cdots\ +\ a_0\ g_n\ =\ e_n \quad n = k, k+1, \cdots$$

def. $\Rightarrow G(z) = \sum_{n=0}^{\infty} g_n z^n$

$$\Rightarrow \sum_{n=k}^{\infty} \sum_{i=0}^{N} \alpha_i g_{n-i} z^n = \sum_{n=k}^{\infty} e_n z^n$$

$\Rightarrow$ carry out summations recognize $G(z)$,
solve for $G(z)$ algebraically,
then invert to get g_n

Example - Use z-Tranform

(same) $$6g_n - 5g_{n-1} + g_{n-2} = 6\left(\frac{1}{5}\right)^n \qquad n = 2, 3, 4, \cdots$$

$$\Rightarrow \sum_{n=2}^{\infty} 6g_n z^n - \sum_{n=2}^{\infty} 5g_{n-1} z^n + \sum_{n=2}^{\infty} g_{n-2} z^n = \sum_{n=2}^{\infty} 6\left(\frac{1}{5}\right)^n z^n$$

$$\Rightarrow 6\sum_{n=2}^{\infty} g_n z^n - 5z\sum_{n=2}^{\infty} g_{n-1} z^{n-1} + z^2 \sum_{n=2}^{\infty} g_{n-2} z^{n-2} = \sum_{n=2}^{\infty} 6\left(\frac{1}{5}\right)^n z^n$$

$$\Rightarrow 6\left[G(z) - g_0 - g_1 z\right] - 5z\left[G(z) - g_0\right] + z^2 G(z) = \frac{6\left(\frac{1}{5}\right)^n z^n}{1 - \left(\frac{1}{5}\right)^z}$$

$$\Rightarrow G(z) = \frac{6g_0 + 6g_1 z - 5g_0 z + \frac{\left(\frac{6}{25}\right)z^2}{1 - \left(\frac{1}{5}\right)z}}{6 - 5z + z^2}$$

$$\Rightarrow \text{using init. cond.:} \quad (g_0 = 0, g_1 = \frac{6}{5})$$

$$G(z) = \left(\frac{1}{5}\right) \frac{z(6-z)}{\left[1 - \left(\frac{1}{3}\right)z\right]\left[1 - \left(\frac{1}{2}\right)z\right]\left[1 - \left(\frac{1}{5}\right)z\right]}$$

Example - Use z-Tranform (Cont...)

part. frac. exp. $\Rightarrow \quad G(z) = \frac{-9}{1-\left(\frac{1}{3}\right)z} + \frac{8}{1-\left(\frac{1}{2}\right)z} + \frac{1}{1-\left(\frac{1}{5}\right)z}$

$$\Rightarrow \quad g_n = -9\left(\frac{1}{3}\right)^n + 8\left(\frac{1}{2}\right)^n + \left(\frac{1}{5}\right)^n \qquad n = 0, 1, 2, \cdots$$

Constant-Coeff. Linear Differential Equations

⇒ ***Nth*** **order eq.:**

$$a_N \frac{d^N f(t)}{dt^N} + a_{N-1} \frac{d^{N-1} f(t)}{dt^{N-1}} + \cdots + a_1 \frac{df(t)}{dt} + a_0 f(t) = e(t)$$

- ⇒ **a_i's are const., *e(t)* is a given func.**
- ⇒ **also given N init. cond. (usually first N derivatives, usually at *t=0*).**
- ⇒ **find *f(t)***

⇒ **have $f^{(h)}(t)$ and $f^{(p)}(t)$**

form ⇒ $f^{(h)}(t) = Ae^{\alpha t}$

subst. ⇒ $a_N A\alpha^N e^{\alpha t} + a_{N-1} A\alpha^{N+1} e^{\alpha t} + \cdots + a_1 A\alpha e^{\alpha t} + a_0 A e^{\alpha t} = 0$

⇒ **has N solutions which must solve char. eq:**

$$a_N \alpha^N + a_{N-1}\alpha^{N+1} + \cdots + a_1\alpha + a_0 = 0$$

⇒ **if all α_i are distinct, then**

$$f^{(h)}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \cdots + A_N e^{\alpha_N t}$$

where A_i's are determined using init. cond.

Constant-Coeff. Linear Differential Equations (Cont...)

$\Rightarrow$ **when have multiple root α_1 of order k, then have**

$$(A_{11}t^{k-1} + A_{12}t^{k-2} + \cdots + A_{1k-1}t + A_{1k})e^{\alpha_1 t}$$

contribute in above form to homogeneous sol.

$\Rightarrow$ **make a guess to find the particular sol. $f^{(p)}(t)$**

complete sol. $\Rightarrow$ $f(t) = f^{(h)}(t) + f^{(p)}(t)$

Example

$$\frac{d^2 f(t)}{dt^2} - 6\frac{df(t)}{dt} + 9f(t) = 2t \qquad (*)$$

$$f(0^-) = 0 \quad \frac{df(0^-)}{dt} = 0 \qquad \text{(init. cond.)}$$

homogen. sol.

$\Rightarrow$ forming characteristic equation:

$$\alpha^2 - 6\alpha + 9 = 0$$

$\Rightarrow$ find multiple root $\alpha_1 = \alpha_2 = 3$

$\Rightarrow$ homogeneous sol. must be of the form

$$f^{(h)}(t) = (A_{11}t + A_{12})e^{3t}$$

$\Rightarrow$ guess

$$f^{(p)}(t) = B_1 + B_2 t$$

$\Rightarrow$ substituting this back into $(*)$ we get

$$B_1 = \frac{4}{27}, \quad B_2 = \frac{2}{9}$$

Example (Cont...)

complete sol. $\Rightarrow \quad f(t) = (A_{11}t + A_{12})e^{3t} + \dfrac{4}{27} + \dfrac{2}{9}t$

using init. cond. $\Rightarrow \quad A_{11} = \dfrac{2}{9}, \quad A_{12} = -\dfrac{4}{27}$

final sol. $\Rightarrow \quad f(t) = \dfrac{2}{9}\left(t - \dfrac{2}{3}\right)e^{3t} + \dfrac{2}{9}\left(t + \dfrac{2}{3}\right) \quad t \geq 0$

Example - Using Laplace Transform Method

(using prev. example) $$\frac{d^2 f(t)}{dt^2} - 6\frac{df(t)}{dt} + 9f(t) = 2t$$

$$s^2F^*(s) - sf(0^-) - f^{(1)}(0^-) - 6sF^*(s) + 6f(0^-) + 9F^*(s) = \frac{2}{s^2}$$

(using init. cond.'s, equal to 0, eliminate some terms above)

$$\Rightarrow \quad F^*(s) = \frac{\frac{2}{s^2}}{s^2 - 6s + 9}$$

part. frac. exp. $$\Rightarrow \quad F^*(s) = \frac{\frac{2}{9}}{s^2} + \frac{\frac{4}{27}}{s} + \frac{\frac{2}{9}}{(s-3)^2} + \frac{-\frac{4}{27}}{s-3}$$

inverting $$\Rightarrow \quad f(t) = \frac{2}{9}t + \frac{4}{27} + \frac{2}{9}te^{3t} - \frac{4}{27}e^{3t}$$

(In queueing theory, sometimes need both, i.e., have differential-difference equations $\Rightarrow$ then use both z and Laplace tranforms).

Deriving Moments Via Transforms

- **Discrete time**

z-Transform of a pmf p_k is given by $G_X(z) = \sum_k p_k \, z^k$

$$\Rightarrow \frac{\partial \, G_X(z)}{\partial \, z} = \sum_k k \, p_k \, z^{k-1}$$

$$\left. \frac{\partial \, G_X(z)}{\partial \, z} \right|_{z=1} = \sum_k k \, p_k = \overline{X}$$

(Expectation)

$$\Rightarrow \frac{\partial^2 \, G_X(z)}{\partial \, z^2} = \sum_k k(k-1) \, p_k \, z^{k-2}$$

$$\left. \frac{\partial^2 \, G_X(z)}{\partial \, z^2} \right|_{z=1} = \sum_k k(k-1) \, p_k = \overline{X^2} - \overline{X}$$

(not quite variance, but can get it from this and the exp.)

$$\Rightarrow \vdots$$

Deriving Moments Via Transforms (Cont...)

- **Continuous time**

Laplace transform of a density func. $f(x)$ is $F^*(s) = \int_0^\infty e^{-sx} \, f(x) \, dx$

$$\Rightarrow \quad \frac{d \; F^*(s)}{d \; s} \; = \; - \int_0^\infty e^{-sx} \; x f(x) \; dx$$

$$\left.\frac{d \; F^*(s)}{d \; s}\right|_{s=0} \; = \; - \overline{X}$$

$$\Rightarrow \quad \left.\frac{d^2 \; F^*(s)}{d \; s^2}\right|_{s=0} \; = \; \overline{X^2}$$

$$\Rightarrow \; \vdots$$

HW: Compute first 2 moments of geometric and exponential distributions using transforms

Relationship of Transforms to Expectations

$$\Rightarrow \; f(t) \;\Leftrightarrow\; F^*(s) \;=\; \int_t e^{-st}\, f(t)\, dt \;=\; E\left[e^{-st}\right]$$

$$\Rightarrow \; p_X(k) \;\Leftrightarrow\; G(z) \;=\; \sum_k p_X(k)\, z^k \;=\; E\left[z^k\right]$$

Inequalities and Limit Theorems

- May not be possible to determine distributions, but might be able to derive and use:
  - (a) moments
  - (b) inequalities and limits

Markov Inequality

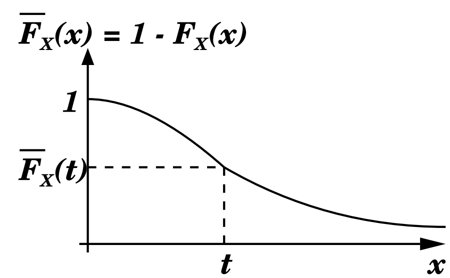
- **Simple Markov Inequality:**
  - $\Rightarrow$ **If only know the expectation, provides a bound on probability distribution**

$\Rightarrow$ **For a r.v. X with mean μ, the Markov Inequality is:**

$$P[X \geq t] \;\leq\; \frac{\mu}{t}$$

assume X is a non-negtive r.v.

<u>Proof</u>:



- **area under the curve**

$$= \int_0^\infty \overline{F_X}(x) \;=\; E[X] \;=\; \mu$$

- **area of rectangle $\leq$ area under the curve**

$$\Rightarrow \overline{F_X}(t) \cdot t \;\leq\; E[X]$$

$$\Downarrow$$

$$P[X \geq t] \cdot t \;\leq\; \mu \;\Rightarrow\; P[X \geq t] \;\leq\; \frac{\mu}{t}$$

Example

- Consider a system with MTTF of *100* hours
- Let X be a r.v. denoting the lifetime of a system
- By Markov Inequality:

$$P[X \geq t] \;\leq\; \frac{\mu}{t} \;=\; \frac{100}{t}$$

- Define reliability of a system as

$$R(t) \;\equiv\; P[X \geq t]$$

- Then

$$R(t) \;\leq\; \frac{100}{t}$$

$\Rightarrow$ To ensure system reliability more than *0.9*, the system mission time $t < 111$.

Example (Cont...)

HW: How tight is the Markov bound for the exponential distribution with parameter

$$\lambda = \frac{1}{100}$$

Review Distribution

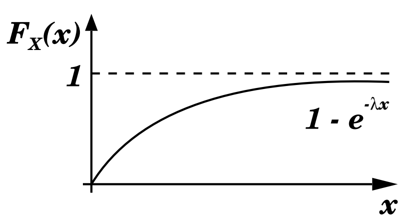
Exponential distribution:

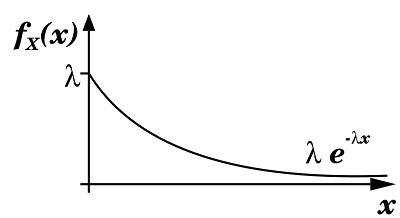
$$P[X \leq x] = 1 - e^{-\lambda x} \qquad x \geq 0$$

$$\Downarrow$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$$f_X(x) = \frac{d\, F_X(x)}{d\, x} = \lambda\, e^{-\lambda x} \qquad x \geq 0$$





Chebycher's Inequality

$\Rightarrow$ **Assume we know μ and σ^2 of r.v. X, then for all $t > 0$,**

$$P[\ |X - \mu| \geq t\ ] \ \leq\ \frac{\sigma^2}{t^2}$$

<u>Proof</u>:

- **let $Y = (X - \mu)^2$ $\Rightarrow$ Y is a non-negative r.v.**
- **applying Markov Inequality:**

$$P\left[\ (X - \mu)^2 \geq t^2\ \right] \ \leq\ \frac{E\left[\ (X-\mu)^2\ \right]}{t^2} \ =\ \frac{\sigma^2}{t^2}$$

but

$$P\left[\ (X - \mu)^2 \geq t^2\ \right] \ =\ P[\ |X - \mu| \geq t\ ]$$

Weak Law of Large Numbers

- **Let** $X_1, \; X_2, \; \cdots, \; X_n$ **be independant identically distributed r.v.'s with** $E\left[X_i\right] = \mu$ and $Var\left[X_i\right] = Var\left[X\right] = \sigma_X^2 \;\; \forall \; i$
- **Define arithmetic mean to be** $\dfrac{X_1 + X_2 + \cdots + X_n}{n}$
- **We would expect that for sufficient large** n

$$\frac{\sum_{i=1}^{n} X_i}{n} \longrightarrow \mu$$

- **Let** $S_n = \sum_{i=1}^{n} X_i$ **; consider r.v.** $\dfrac{S_n}{n}$

$$\Rightarrow \; E\left[\frac{S_n}{n}\right] = \frac{n\,\mu}{n} = \mu$$

$$\Rightarrow \; Var\left[\frac{S_n}{n}\right] = \frac{1}{n^2} Var\left[\, S_n \,\right] = \frac{1}{n^2}\, n\, \sigma_X^2 = \frac{\sigma_X^2}{n}$$

$$\text{so, as } n \longrightarrow \infty \qquad Var\left[\frac{S_n}{n}\right] \longrightarrow 0$$

Weak Law of Large Numbers (Cont...)

- **Applying Chebychev's Inequality to $\frac{S_n}{n}$, we get**

$$P\left[\;\left|\frac{S_n}{n} - \mu\right| \geq \delta\;\right] \;\leq\; \frac{\sigma^2}{n\;\delta^2}$$

$$\Rightarrow\; P\left[\;\left|\frac{S_n}{n} - \mu\right| \geq \delta\;\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

$\Rightarrow$ **the distribution of arithmetic mean $\frac{S_n}{n}$ becomes increasingly concentrated around the mean μ as n grows!**

$\Rightarrow$ **δ can be thought of as the error in approximating μ by the arithmetic mean**

$$\lim_{n\to\infty} P\left[\;\left|\frac{S_n}{n} - \mu\right| \geq \delta\;\right] \longrightarrow 0 \;\longleftarrow \textbf{weak law of large numbers}$$

or

$$\lim_{n\to\infty} P\left[\;\left|\frac{S_n}{n} - \mu\right| < \delta\;\right] \longrightarrow 1$$

$\Rightarrow$ **for any small δ, as n grows, the error will be less than δ with probability *1***

Strong Law of Large Numbers

$$\Rightarrow \quad \lim_{n\to\infty} \frac{S_n}{n} \;=\; E[X] \;=\; \mu \longrightarrow P\left[\; \lim_{n\to\infty} \left|\frac{S_n}{n} - E[X]\right| \geq \delta \;\right] \;=\; 0$$

$$\text{or } \frac{S_n}{n} \xrightarrow{\text{a.s.}} E[X] \;\; \text{as } n \longrightarrow \infty$$

Central Limit Theorem

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\overline{X}}{\sigma_X \sqrt{n}}$$

where X_i are i.i.d.
with mean $\overline{X}$ and variance σ_X^2

PDF of Z_n tends to the standard normal, i.e., **x is real**

$$P_{n\to\infty}[\ Z_n \leq x\ ] = \phi(x)$$ **where** $$\phi(x) \equiv \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

(Gaussian or normal distribution)